Tests for Perfect Ranking in Moving Extreme Ranked Set Sampling

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Program: Statistics

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Asmaa Taha Ababneh

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بسم الله الرحمن الرحيم

Abstract

Ababneh, Asmaa Taha. Tests for Perfect Ranking in Moving Extreme Ranked Set Sampling. Master of Science Thesis, Department of Statistics, Yarmouk University, 2011. (Supervisor: Prof. Mohammad Fraiwan Al-Saleh).

In this thesis, we take a close look at Moving Extreme Ranked Set Sampling (MERSS) method. The focus is on testing for perfect and imperfect ranking. Several tests are investigated in the case of one and multi-cycle MERSS. All tests are nonparametric and the test statistic of each of them is based on the distance between the observed actual ordering of MERSS (one cycle) and the most likely one under perfect ranking, namely $(1,2,3,\ldots m)$. One of these tests is the well – known Chi-square test. A formula for each possible ordering is derived under the assumption of perfect ranking; the formula turned out to be independent of the underlying distribution. The rejection region is obtained for selected values of set size and the power is calculated for each test for specific alternatives. A real available data set and an artificial bivariate normal data are used for illustration.

Key words: Ranked Set Sampling, Moving Extreme Ranked Set Sampling, Perfect Ranking, Concomitant Variable, Test for Perfect Ranking.

الملخص

عبابنه، أسماء طه. اختبارات لفحص الترتيب الكامل في طريقة المعاينة المرتبة المتحركة. رسالة ماجستير في قسم الإحصاء بجامعة اليرموك، 2011.(المشرف: الأستاذ الدكتور محمد فريوان الصالح)

في هذه الأطروحة، ألقينا الضوء على طريقة المعاينة المرتبة المتحركة "MERSS". كان التركيز على عرض بعض الاختبارات لاختبار فيما إذا كان الترتيب صحيح (عدم وجود خطأ في الترتيب) أو غير صحيح (وجود خطأ في الترتيب). عدة اختبارات تم عرضها في حالة دورة واحدة و في حالة دورات متعددة, كل هذه الاختبارات غير معلميه و تعتمد على المسافة بين الترتيب الحقيقي لعينة "MERSS" وبين الترتيب الكامل (1,2,..,m). وُجد أن الصيغة الرياضية للترتيب المتوقع التي اشتقت تحت فرضية أن الترتيب صحيح هي صيغة لا تعتمد على أي توزيع. عند تحديد حجم للعينة استطعنا إيجاد منطقة الرفض لهذه القيمة، أيضاً تم إيجاد قوة الاختبار لكل الاختبارات المقترحة. و للتوضيح تم عرض بعض الأمثلة.

الكلمات المفتاحية: المعاينة المرتبة، المعاينة المرتبة المتحركة، الترتيب الصحيح، المتغير المصاحب، اختبارات للترتيب الصحيح.

Chapter 1

Introduction and Literature Review

1.1. Sampling Techniques

Statistics is the science of making inference about a population using the information in a chosen sample. The branch of statistics that deals with the description and summarization of the sample data is called descriptive statistics and that deals with the inference is called inferential statistics. The precision and / or accuracy of the conclusions depend on how representative is the sample of the population.

There are many sampling techniques that can be used to choose a suitable sample from a population. The most famous techniques are:

- **Simple Random Sampling**

SRS is the basic method for all other sampling techniques. In sampling for finite populations, SRS is a method of choosing a sample of size n $(n \geq 1)$ from a population of size N so that all subsets of size n have equal chances of being selected.

- **Cluster Random Sampling**

This method is basically a simple random sample in which each sampling unit is a collection or cluster of elements. It is suitable when there is no frame for the elements of the population but there is a frame for the collection of units (clusters).

- **Systematic Random Sampling**

A systematic sample can be obtained by selecting at random one element from the first k elements in the frame and every kth element thereafter; it is called a $1 - in - k$ systematic sample. Usually k is taken to be $\left[\frac{N}{n}\right]$.

- **Stratified Random Sampling**

A stratified sample can be obtained by stratifying (dividing) the population elements into non overlapping groups; each group called a stratum, and then take a random sample (SRS, Systematic, etc.) from each stratum separately. If stratification of the population is not easy before taking the sample, Post stratification of the chosen sample can have some advantages. (For more information of these technique see "Elementary Survey Sampling" by Scheaffer, Mendenhall, & Ott 1986).

- **Ranked Set Sampling**

McIntyre (1952) introduced a new sampling technique called ranked set sampling to estimate more effectively yields of pastures. This method is suitable for situation when the units can be ranked (with respect to the variable of interest) by judgment without actual measurement.

The main idea of RSS is similar to stratified sampling; in stratified sampling, the population is divided by judgment into sub populations (strata) such that elements are more similar within strata than among strata. In ranked set sampling, we are trying to do the same as in stratified sampling but at the level of the sample rather than the level of the population.

The RSS technique can be executed as follows:

1. Draw randomly m sets of size m each from the population of interest.
2. The elements within each set are ranked by judgment, without doing actual quantification, with respect to the variable of interest from smallest to largest. It is assumed here that each element can be ranked by eyes or by a relatively cheap method.
3. From the i^{th} set, take for actual quantification the element (judgment) ranked as the i^{th} order statistic. This cycle yields a sample of size m.
4. The above procedure can be repeated r times to get a sample of size $n = rm$.

One problem with RSS procedure is the requirement that the ranking should be done by judgment or at a negligible extra cost. Thus, it is usually hard to believe that the ranking is perfect. Error in ranking is usually unavoidable especially with large set size m. Al-Odat and Al-Saleh (2001) introduced a modified procedure that only identifies the extreme judgment order statistics of sets of varied size. The procedure was further investigated by Al-Saleh and Al-Hadhrami (2003 a, b). It coined by them as moving extreme ranked set sampling (MERSS) procedure. In this thesis, the focus will be on this modification. In particular, we are interested in investigating error in ranking in MERSS.

1.2. Literature Review

The first one who proposed the RSS was McIntyre (1952) as mentioned previously. He showed that RSS is a suitable method for situations when the elements of population can be ranked by visual inspection or by any cheap method. He showed without proof that (i) the mean of the quantified elements is an unbiased estimator of the population mean regardless of any error in judgment ranking and (ii) with perfect ranking for typical unimodal distribution; the mean of such a sample is nearly $(m+1)/2$ times as efficient as the mean of a simple random sample of the same size.

In (1966) a second paper was published on RSS by Halls and Dell, in this paper they studied the performance of ranked set sample in estimating the weights of browse and of herbage in a pine-hardwood forest.

Takahasi and Wakimoto (1968) established a statistical theory of ranked set sample. The authors arrived at the method independently of McIntyre. The method is proposed when (i) Estimation of a population mean is of primary interest. (ii) Acquisition of sampling units is cheap compared with their quantification. (iii) A small set of sampling units can be easily ranked with respect to the character of interest without quantifying the units, with the assumption that the ranking in (iii) can be done without error.

The RSS estimator of the population mean μ according to Takahasi and Wakimoto works can be described as follows: let T_i be the sum of the quantifications of the i^{th} ranked units for $i = 1,2,3,\ldots\ldots,m$. Let $r_1, r_2, r_3, \ldots\ldots, r_m$ be positive integers such that $\sum_{i=1}^{m} r_i = n$ then,

$$\hat{\mu}_{RSS} = \frac{1}{m}\sum_{i=1}^{m}\frac{T_i}{r_i}.$$

Assume the population has density function $f(x)$, mean μ and variance σ^2. Let the $(i{:}m)^{th}$ order statistic from the population have density function $f_{i:m}(x)$, mean $\mu_{i:m}$, and variance $\sigma_{i:m}^2$. Takahasi and Wakimoto basic identity is:

$$f(x) = \frac{1}{m}\sum_{i=1}^{m} f_{i:m}(x),$$

and the mean and the variance are:

$$\mu = \frac{1}{m}\sum_{i=1}^{m} \mu_{i:m},$$

$$\sigma^2 = \frac{1}{m}\sum_{i=1}^{m} \sigma_{i:m}^2 + \frac{1}{m}\sum_{i=1}^{m}(\mu_{i:m} - \mu)^2.$$

Also, Takahasi and Wakimoto showed that the mean of RSS is the best linear unbiased estimator of the population mean and has higher efficiency than the mean of SRS with the same number n of quantifications. For more details and results on RSS technique see Kaur et al. (1995), and Chen et al. (2004).

Recently, tests for perfect ranking in RSS was introduced by Li and Balakrishnan (2008) and Vock and Balakrishnan (2011). They proposed nonparametric tests for the assumption of perfect ranking. Li and Balakrishnan defined three tests based on one-cycle RSS. Also, they derived the exact null distributions and exact power functions of all these tests. Their proposed tests were based on the probability of $P(Y_{i_1} \leq Y_{i_2} \leq \cdots \leq Y_{i_m})$, where Y_{i_j} is the i_j^{th} order statistic of a sample of size m, which was obtained by Al-Saleh and Al-Kadiri (2000). Also, other tests were proposed by Li and Balakrishnan (2008) based on multi-cycle RSS and the performance of all these tests were compared with that of Kolmogorove -Smirnov test statistic. In addition, Vock and Balakrishnan used the test statistic that formally corresponds to the Jonckheere-Terpstra-type test.

Al-Odat and Al-Saleh (2001) introduced a new modified technique of RSS; later it was coined by Al-Saleh and Al-Hadhrami (2003) as "Moving Extreme Ranked Set Sampling" (MERSS). They showed that this modification of RSS can be more useful than SRS and easier to perform.

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They investigated this method nonparametrically and concluded that the estimator of the population mean is more efficient than that of SRS in the case of symmetric populations.

The method was considered parametrically under exponential distribution by Al-Saleh and Al-Hadhrami (2003 a, b); they studied this method in case of perfect and imperfect ranking. Also, the maximum likelihood estimator (MLE) and modified MLE of the population mean were considered. They concluded that the MLE of the mean of the exponential distribution based on MERSS is more efficient than the MLE based on SRS. Also, the information contained in MERSS, measured by Fisher information number, is always greater than that of SRS with the same size.

Al-Saleh and Al-Ananbeh (2005) considered the estimation of correlation coefficient in the bivariate normal distribution based on MERSS using a concomitant random variable. It was concluded that MERSS with concomitant variable is a useful modification of RSS to estimate the correlation coefficient.

Also, Al-Saleh and Al-Ananbeh (2007) considered the estimation of the means of the bivariate normal distribution based on MERSS with concomitant variable. It appeared that the suggested estimator is more efficient.

Abu-Dayyeh and Al-Sawi (2009) made inference about the scale parameter of the exponential density in the case of MERSS by using the maximum likelihood estimator and the likelihood ratio test (LRT). Because there is no closed form for the LRT and MLE, they used the modified MLE to come up with a modified LRT to test a simple hypothesis against one sided alternative.

Most recently, Al-Saleh and Samawi (2010) used the MERSS to estimate the odd of CDF, $F/(1-F)$. The suggested estimator is motivated by some of the theoretical properties of the sum of geometric series. They compared the performance of this estimator with the estimator based on SRS. It turned out that the estimator based on MERSS is always valid and can have some advantages over that based on SRS.

Currently, MERSS is being considered by Hanandeh (2011) for the estimation of the parameters of Downton's distribution.

1.3. The procedure of Moving Extreme Ranked Set Sampling

The MERSS technique can be described as follows:

1) Select m simple random samples of size $1,2,3,\ldots,m$, respectively.
2) Measure accurately the maximum ordered observation from the first set, the maximum ordered observation from the second set; the process continues in this way until the maximum order observation from the last m^{th} sample is measured.
3) Step (2) may be repeated if needed on another m samples of size $1,2,3,\ldots,m$ respectively, but here the minimum ordered observations are measured instead of the maximum ordered observations.
4) Steps (1-3) can be repeated, if necessary, many times to obtain a sample of larger size.

In this work, we will only consider the maximum i.e. step (1-2, 4).

1.4. Organization of the thesis

In this thesis, we are going to investigate the MERSS technique non-parametrically; i.e. there is no assumption that the distribution is known. In Chapter (2), the probability of $P(Y_{[i_1:i_1]} \leq Y_{[i_2:i_2]} \leq \cdots \leq Y_{[i_m:i_m]})$ under perfect and imperfect ranking are derived, and some properties are listed and proved. Then, three simple non-parametric tests are investigated to test for perfect ranking for one-cycle MERSS. For specific values of m; the exact null distributions of these tests are found, and the exact power functions under some specific alternatives are derived. In Chapter (3), tests that deal with multi-cycle MERSS are introduced. Samples from bivariate normal distribution are used for illustrations. In Chapter (4), general conclusions and suggested future works are presented.

Chapter 2

Test for Perfect and Imperfect Ranking in MERSS-One Cycle

2.1. Introduction

In this chapter, we will consider the error in ranking in MERSS. Three statistical tests to test for imperfect ranking will be used. The three tests are denoted by N_m, S_m and A_m. The three tests are investigated based on one cycle MERSS. In Section 2.2, basic terminology are given. In Section 2.3, the formula of the term $\pi(i_1, i_2, i_3, \dots\dots, i_m) = P(Y_{[i_1:i_1]} \leq Y_{[i_2:i_2]} \leq \cdots \leq Y_{[i_m:i_m]})$ under perfect ranking will be derived; also some properties will be listed and proved. Then in Section 2.4, the same probability will be derived under error in ranking. In Section 2.5, the three tests will be introduced. Tables for critical values of the tests and the power comparison are given in Section 2.6. A real data example is discussed in Section 2.7. Concluding remarks are given in Section 2.8.

2.2. Basic Terminology

Let $\{Y_{[i:i]}\ i = 1,2,\dots,m\}$ be a MERSS of size m, where $Y_{[i:i]}$ is the judgment maximum order statistic of a SRS of size i from a population with pdf $f(x)$ and CDF $F(x)$; $F(x)$ is assumed to be absolutely continuous. Also, assume that $Y_{(i:i)}$ is the actual maximum order statistic of a SRS of size i. The probability density functions of $Y_{(i:i)}$ and $Y_{[i:i]}$, respectively, are:

$f_{(i:i)}(y) = i(F(y))^{i-1}f(y), \qquad i = 1,2,\dots,m.$

$f_{[i:i]}(y) = \sum_{k=1}^{i} a_{ki} f_{(k:i)}(y)$. (For more information about this formula see C. Frey (2007)) where $a_{ki} \geq 0$, $\sum_{k=1}^{i} a_{ki} = 1$, and

$$f_{(k:i)}(y) = i\binom{i-1}{k-1}\left((F(y))^{k-1}(1-F(y))^{i-k}f(y). \quad i = 1,2,\ldots m, k = 1,2,\ldots i.$$

a_{ki} can be thought of as the probability that the $Y_{[i:i]}$ has the density $f_{(k:i)}(y)$.

Note: If the ranking is perfect, then $a_{ki} = 1$ for $k = i$ and $a_{ki} = 0$ otherwise.

Let $\pi(i_1, i_2, i_3, \ldots\ldots, i_m) = P(Y_{[i_1:i_1]} \leq Y_{[i_2:i_2]} \leq \cdots \leq Y_{[i_m:i_m]})$. In this chapter our hypotheses are:

H_0: Ranking is perfect (no ranking error).

H_1: There is some ranking error (ranking is not perfect).

i.e. H_0: $a_{ki} = 1$ for $k = i$ and $a_{ki} = 0$ for $k \neq i$, where $i = 1,2,\ldots m$. H_1: H_0 is not true.

2.3. $\pi(i_1, i_2, i_3, \ldots\ldots, i_m)$ Under Perfect Ranking

Let $Y_{[1:1]}, Y_{[2:2]}, \ldots\ldots Y_{[m:m]}$ be a MERSS of size m, let $(i_1, i_2, i_3, \ldots\ldots, i_m)$ be any permutation of $(1,2,\ldots\ldots, m)$ and let $\pi(i_1, i_2, i_3, \ldots\ldots, i_m) = P\left(Y_{[i_1:i_1]} \leq Y_{[i_2:i_2]} \leq \cdots \leq Y_{[i_m:i_m]}\right)$. The value of $\pi(i_1, i_2, i_3, \ldots\ldots, i_m)$ under perfect ranking is given in the following theorem.

Theorem (1): For any permutation $(i_1, i_2, \ldots\ldots, i_m)$ of $(1,2, \ldots\ldots, m)$, if H_0 is true then:

$$\pi_0(i_1, i_2, i_3, \ldots\ldots, i_m) = \frac{m!}{\prod_{k=1}^{m} \sum_{j=1}^{k} i_j} \quad \ldots\ldots(1)$$

For simplicity, we will use Y_{i_k} instead of $Y_{[i_k:i_k]}$.

Proof: Since $Y_{[1:1]}, Y_{[2:2]}, \ldots, Y_{[m:m]}$ are independent,

$\pi_0(i_1, i_2, i_3, \ldots\ldots, i_m) =$

$\int_{-\infty}^{\infty}\int_{-\infty}^{y_{i_m}} \ldots\ldots\ldots \int_{-\infty}^{y_{i_3}}\int_{-\infty}^{y_{i_2}} f_{(1:1)}(y_{i_1}) f_{(2:2)}(y_{i_2}) \ldots\ldots. f_{(m:m)}(y_{i_m}) dy_{i_1} dy_{i_2} \ldots\ldots dy_{i_m}$

Since $Y_{i_j} \sim f_{(i_j:i_j)}(y) = i_j \left(F(y)\right)^{i_j-1} f(y)$; for $j = 1,2, \ldots\ldots, m$.

$\pi_0(i_1, i_2, i_3, \ldots\ldots, i_m) =$

$i_1 i_2 \cdots i_m \int_{-\infty}^{\infty}\int_{-\infty}^{y_{i_m}} \cdots \int_{-\infty}^{y_{i_3}}\int_{-\infty}^{y_{i_2}} f(y_{i_1}) \left(F(y_{i_1})\right)^{i_1-1} . f(y_{i_2}).(F(y_{i_2}))^{i_2-1} \ldots f(y_{i_m}) (F(y_{i_m}))^{i_m-1}\, dy_{i_1} dy_{i_2} \ldots dy_{i_m}$

Let $F(y_{i_j}) = u_{i_j}$, then $f\left(y_{i_j}\right) dy_{i_j} = du_{i_j}$, where $j = 1,2, \ldots m$. Thus,

$$\pi_0(i_1, i_2, i_3, \ldots\ldots, i_m) = m! \int_0^1 \int_0^{u_{i_m}} \ldots\ldots \int_0^{u_{i_2}} \prod_{j=1}^{m} u_{i_j}^{i_j-1}\, du_{i_j}$$

$$= m! \int_0^1 \int_0^{u_{i_m}} \ldots\ldots \left(\int_0^{u_{i_2}} u_{i_1}^{i_1-1} du_{i_1}\right) \prod_{j=2}^{m} u_{i_j}^{i_j-1}\, du_{i_j}$$

$$= \frac{m!}{i_1} \int_0^1 \int_0^{u_{i_m}} \ldots\ldots\ldots \left(\int_0^{u_{i_3}} u_{i_2}^{i_1} \,. u_{i_2}^{i_2-1}\, du_{i_2}\right) \prod_{j=3}^{m} u_{i_j}^{i_j-1}\, du_{i_j}$$

$$= \frac{m!}{i_1(i_1+i_2)} \int_0^1 \ldots\ldots \left(\int_0^{u_{i_4}} u_{i_3}^{i_1+i_2}\, u_{i_3}^{i_3-1}\, du_{i_3}\right) \prod_{j=4}^{m} u_{i_j}^{i_j-1}\, du_{i_j}$$

$$= \frac{m!}{i_1(i_1+i_2)(i_1+i_2+i_3)} \int_0^1 \ldots\ldots \left(\int_0^{u_{i_5}} u_{i_4}^{i_1+i_2+i_3}\, u_{i_4}^{i_4-1}\, du_{i_4}\right) \prod_{j=5}^{m} u_{i_j}^{i_j-1}\, du_{i_j}$$

$$= \frac{m!}{i_1(i_1+i_2)(i_1+i_2+i_3)(i_1+i_2+i_3+i_4)} \int_0^1 \ldots\ldots \left(\int_0^{u_{i_6}} u_{i_5}^{i_1+i_2+i_3+i_4}\, u_{i_5}^{i_5-1}\, du_{i_5}\right) \prod_{j=5}^{m} u_{i_j}^{i_j-1}\, du_{i_j}$$

$$= \frac{m!}{i_1 * (i_1+i_2) * \ldots * (i_1+i_2+\cdots+i_m)}$$

i.e.

$$\pi_0(i_1, i_2, i_3, \ldots\ldots, i_m) = \frac{m!}{\prod_{k=1}^{m} \sum_{j=1}^{k} i_j} \blacksquare$$

A special case of $\pi_0(i_1, i_2, i_3, \ldots\ldots, i_m)$ is given in the following lemma.

Lemma (1): Assume that $(i_1, i_2, i_3, \ldots\ldots, i_m) = (1,2, \ldots\ldots, m)$, then:

$$\boldsymbol{\pi_0(1, 2, 3, \ldots\ldots, m) = \frac{2^m}{(m+1)!}} \qquad \ldots\ldots(2)$$

Proof:

Substitute $(1,2,3, \ldots\ldots, m)$ in $\pi_0(i_1, i_2, i_3, \ldots\ldots, i_m) = \frac{m!}{i_1(i_1+i_2)+\cdots(i_1+i_2+\cdots+i_m)}$ to get:

$$\pi_0(1,2,3, \ldots\ldots, m) = \frac{m!}{1 * (1+2) * (1+2+3) * \ldots * (1+2+3+\cdots+m)}$$

This can be written as:

$$\pi_0(1,2,3, \ldots\ldots, m) = \frac{m!}{\prod_{k=1}^{m} k(k+1)/2} = \frac{2^m\, m!}{m!\,(m+1)!} = \frac{2^m}{(m+1)!} \blacksquare$$

Properties of $\boldsymbol{\pi_0(i_1, i_2, i_3, \ldots\ldots, i_m)}$:

Suppose that the ranking is perfect, then we have the following properties:

1) For $i_1 < i_2$, $P(Y_{i_1} < Y_{i_2}) > 0.5$.
2) $\pi_0(1,2,3, \ldots\ldots, m) \geq \frac{1}{m!}$.
3) $\pi_0(i_1, i_2, i_3, \ldots\ldots, i_m) \leq \pi_0(1,2,3, \ldots\ldots, m)$, i.e. the maximum value of π_0 occurred at $(1,2, \ldots, m)$.

4) $\pi_0(i_1, i_2, i_3, \ldots\ldots, i_m) \geq \pi_0(m, m-1, \ldots\ldots, 2,1) = \frac{2^m m!}{(2m)!}$ i.e. the minimum value of π_0 occurred at $(m, m-1, \ldots\ldots, 2,1)$.

Proof:

1. Suppose $i_1 < i_2$.

$$P(Y_{i_1} < Y_{i_2}) = \int_{-\infty}^{\infty}\int_{-\infty}^{y_{i_2}} f_{(1:1)}(y_{i_1}) f_{(2:2)}(y_{i_2})\ dy_{i_1} dy_{i_2}$$

$$= i_1 * i_2 \int_{-\infty}^{\infty}\int_{-\infty}^{y_{i_2}} f(y_{i_1})\ \left(F(y_{i_1})\right)^{i_1-1} f(y_{i_2})(F(y_{i_2}))^{i_2-1}\ dy_{i_1} dy_{i_2}$$

$$= i_1 * i_2 \int_0^1 \int_0^{u_{i_2}} u_{i_1}^{i_1-1} u_{i_2}^{i_2-1} du_{i_1} du_{i_2}$$

$$= \frac{i_1 * i_2}{i_1} \int_0^1 u_{i_2}^{i_1+i_2-1} du_{i_2} = \frac{i_2}{i_1+i_2} ,$$

which is larger than 0.5 because:

$$\frac{i_2}{i_1+i_2} > \frac{1}{2} \text{ iff } 2i_2 > (i_1+i_2) \text{ iff } i_1 < i_2.$$

2. Property (2) will be proved by induction:

- For $m = 2$, $\pi_0(1,2) = \frac{2!}{1*(1+2)} = \frac{2}{3} \geq \frac{1}{6}$ (true).
- Suppose it is true for $m = k$, i.e. $\pi_0(1,2,3, \ldots\ldots, k) = \frac{2^k}{(k+1)!} \geq \frac{1}{k!}$
- Now,

$$\pi_0(1,2,3, \ldots\ldots, k, k+1) = \frac{2^{k+1}}{(k+2)!}$$

$$= \frac{2\,2^k}{(k+2)(k+1)!} \geq \frac{2}{(k+2)}\frac{1}{k!} \geq \frac{1}{(k+1)}\frac{1}{k!} = \frac{1}{(k+1)!} \blacksquare$$

$\frac{2}{k+2} \geq \frac{1}{k+1}$ since $2k+2 \geq k+2 \rightarrow 2k \geq k$, which is true $\forall k$.

3. $\pi_0(i_1, i_2, i_3, \ldots\ldots, i_m) = \frac{m!}{i_1(i_1+i_2)\ldots(i_1+i_2+\cdots i_m)}$

This value attains its maximum when each term of $i_1, i_1 + i_2, \ldots, i_1 + i_2 + \cdots + i_m$ is as small as possible. This occurs when:

$$(i_1, i_2, i_3, \ldots\ldots, i_m) = (1,2,3, \ldots.., m),$$

which gives,

$$\pi_0(1,2,3, \ldots.., m) = \frac{m!}{\prod_{k=1}^{m} k(k+1)/2} = \frac{m!}{\frac{1}{2^m}\, m!\,(m+1)!} = \frac{2^m}{(m+1)!} \blacksquare$$

4. $\pi_0(i_1, i_2, i_3, \ldots\ldots, i_m) = \frac{m!}{i_1 * (i_1+i_2) * \ldots (i_1+i_2+\cdots i_m)}$

$\pi_0(i_1, i_2, i_3, \ldots\ldots, i_m)$ is smallest when each term in the denominator is as large as possible, i.e.:

$$\pi_0(i_1, i_2, i_3, \ldots\ldots, i_m) = \frac{m!}{m * (m+m-1) * \ldots (m+m-1+\cdots+2+1)}$$

$$= \frac{m!}{m(2m-1)(3m-3) * (4m-6)(5m-10)(6m-15)(7m-21)\ldots(m+m-1+\cdots+2+1)}$$

$$= \frac{m!}{\prod_{i=1}^{m}(m+m-i+1) * (\frac{i}{2})}$$

$$= \frac{m!}{\prod_{i=1}^{m}(2m-i+1) * (\frac{i}{2})} = \frac{2^m}{\prod_{i=1}^{m}(2m+1-i)}$$

$$= \frac{2^m}{(2m)!/_{m!}}$$

$$\therefore \ \pi_0(i_1, i_2, i_3, \ldots\ldots, i_m) = \frac{2^m m!}{(2m)!} \qquad \ldots\ldots (3)$$

From (2) and (3) we have:

$$\frac{2^m m!}{(2m)!} \leq \pi_0(i_1, i_2, i_3, \ldots\ldots, i_m) \leq \frac{2^m}{(m+1)!}.$$

2.4. $\pi(i_1, i_2, i_3, \ldots\ldots, i_m)$ Under Imperfect Ranking

In the previous section, we found the value of $\pi(i_1, i_2, i_3, \ldots\ldots, i_m)$ under perfect ranking. Here, we assume that there is an error in judgment ranking; in this case we assume that the probability density function of $Y_{[i:i]}$ take the form:

$f_{[i:i]}(y) = \sum_{k=1}^{i} a_{ki}\, f_{(k:i)}(y),\ \ i = 1,2, \ldots\ldots, m,$ (Frey, 2007)

where, $f_{(k:i)}(y) = i\binom{i-1}{k-1} F(y)^{k-1}\left(1-F(y)\right)^{i-k} f(y),$ for $k = 1,2, \ldots i.$ and $\sum_{k=1}^{i} a_{ki} = 1.$

Now,

$\pi_1(i_1, i_2, i_3, \ldots\ldots, i_m) = P\left(Y_{[i_1:i_1]} \leq Y_{[i_2:i_2]} \leq \cdots \leq Y_{[i_m:i_m]}\right)$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{y_{i_m}} \ldots\ldots\ldots \int_{-\infty}^{y_{i_3}}\int_{-\infty}^{y_{i_2}} f_{[1:1]}(y_{i_1}) f_{[2:2]}(y_{i_2}) \ldots\ldots . f_{[m:m]}(y_{i_m}) dy_{i_1} dy_{i_2} \ldots\ldots dy_{i_m}$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{y_{i_m}} \ldots\ldots \int_{-\infty}^{y_{i_3}}\int_{-\infty}^{y_{i_2}} \prod_{i=1}^{m}\sum_{k=1}^{i} a_{ki} f_{i_k}(y_{i_k})\, dy_{i_k}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{y_{i_m}} \cdots\cdots \int_{-\infty}^{y_{i_3}}\int_{-\infty}^{y_{i_2}} \prod_{i=1}^{m}\sum_{k=1}^{i} a_{ki}\, i \binom{i-1}{k-1} \left(F(y_{i_k})\right)^{k-1} \left(1-F(y_{i_k})\right)^{i-k} f(y_{i_k})\, dy_{i_k}$$

Let $F(y_{i_k}) = u_{i_k}$, $f(y_{i_k})\, dy_{i_k} = du_{i_k}$, then,

$$\pi_1(i_1, i_2, i_3, \ldots\ldots, i_m) = \int_0^1\int_0^{u_{i_m}} \cdots\cdots \int_0^{u_{i_3}}\int_0^{u_{i_2}} \prod_{i=1}^{m}\sum_{k=1}^{i} a_{ki}\, i \binom{i-1}{k-1} u_{i_k}^{k-1}(1-u_{i_k})^{i-k}\, du_{i_k}$$

$$\therefore \pi_1(i_1, i_2, i_3, \ldots\ldots, i_m) = \int_0^1 \cdots\cdot \int_0^{u_{i_3}}\int_0^{u_{i_2}} \prod_{i=1}^{m}\sum_{k=1}^{i} a_{ki}\, i \binom{i-1}{k-1} u_{i_k}^{k-1}(1-u_{i_k})^{i-k}\, du_{i_k} \quad ..(4)$$

Note that for a specific value of m and $a's$, Equation (4) can be easily computed.

For example, for $m = 3$ and $a_{11} = 1$ we have:

$$\pi_1(1,2,3) = \frac{1}{15} a_{12}a_{13} + \frac{1}{6}a_{12}a_{23} + \frac{1}{30}a_{22}a_{13} + \frac{4}{15}a_{12}a_{33} + \frac{2}{15}a_{22}a_{23} + \frac{1}{3}a_{22}a_{33}.$$

Take $a_{12} = \frac{4}{5}, a_{22} = \frac{1}{5}, a_{13} = \frac{8}{10},\ a_{23} = \frac{1}{10}$, and $a_{33} = \frac{1}{10}$, to get: $\pi_1(1,2,3) = 0.092$.

Note:

Under H_0 (perfect ranking),

$$a_{ki} = \begin{cases} 1 & \text{for } k = i \\ 0 & o.w \end{cases}.$$

Which implies that $\pi(i_1, i_2, i_3, \ldots\ldots, i_m)$ is as given in Theorem (1). Conversely if $\pi(i_1, i_2, i_3, \ldots\ldots, i_m) = \pi_0(i_1, i_2, i_3, \ldots\ldots, i_m)$ (as given in Theorem (1)), then $a_{ki} = 1$ for $k = i$ and zero otherwise. This can be seen easily when $m = 2,3$ based on formulas (1) and (4):

Note that for any m, $a_{11} = 1$, for $m = 2$, the equations in the matrix form are:

$$\begin{pmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

We used scientific workplace to solve the above system of equations; the solution is:

$$a_{12} = 0 \text{ and } a_{22} = 1.$$

For $m = 3$, the equations in the matrix form are:

$$\begin{pmatrix} 1/15 & 1/6 & 1/30 & 4/15 & 2/15 & 1/3 \\ 1/10 & 1/10 & 1/5 & 1/20 & 3/10 & 1/4 \\ 1/12 & 7/30 & 1/60 & 13/30 & 1/15 & 1/6 \\ 1/4 & 3/10 & 1/20 & 1/5 & 1/10 & 1/10 \\ 1/6 & 1/15 & 13/30 & 1/60 & 7/30 & 1/12 \\ 1/3 & 2/15 & 4/15 & 1/30 & 1/6 & 1/15 \end{pmatrix} \begin{pmatrix} a_{12}a_{13} \\ a_{12}a_{23} \\ a_{22}a_{13} \\ a_{12}a_{33} \\ a_{22}a_{23} \\ a_{22}a_{33} \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/4 \\ 1/6 \\ 1/10 \\ 1/12 \\ 1/15 \end{pmatrix}$$

The solution is:

$$\begin{pmatrix} a_{12}a_{13} \\ a_{12}a_{23} \\ a_{22}a_{13} \\ a_{12}a_{33} \\ a_{22}a_{23} \\ a_{22}a_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus, since $0 \leq a_{ki} \leq 1$, we have:

$$a_{22} = 1 \, , a_{33} = 1$$

$$\text{and } a_{ki} = 0 \; o.w.$$

The above result can be obtained for larger m. We strongly believe that the result is true for general m, but showing that is getting more tedious as m gets large. We conjecture that:

$$H_0: \pi_0(i_1, i_2, i_3, \ldots\ldots, i_m) = \frac{m!}{\prod_{k=1}^{m} \sum_{j=1}^{k} i_j}.$$

$$H_1: \pi_0(i_1, i_2, i_3, \ldots\ldots, i_m) \neq \frac{m!}{\prod_{k=1}^{m} \sum_{j=1}^{k} i_j}, \text{ for some } (i_1, i_2, i_3, \ldots\ldots, i_m),$$

are equivalent to:

$$H_0: a_{ki} = \begin{cases} 1 & \text{for } k = i \\ 0 & o.w \end{cases}. \quad \text{Versus } H_1: H_0 \text{ is not true.}$$

2.5. Test Statistics

In this section, some simple nonparametric tests will be introduced, these tests can be used to check if the ranking is perfect or not for one cycle MERSS. Under perfect ranking; the value of $\pi(i_1, i_2, \ldots. i_m)$ gets larger as the distance between $(i_1, i_2, \ldots\ldots i_m)$ and $(1,2, \ldots., m)$ gets smaller and vise versa. That means, a suitable test statistic may be based on the distance between these two vectors. The smaller is the distance, the stronger is the evidence that H_0 is true and vise versa.

The following three tests are analogue of the tests that were used by Li and Balakrishnan (2008) for one cycle RSS:

a. N_m: It is the number of inversions in the vector $(i_1, i_2, \ldots\ldots i_m)$, wherein an inversion is the presence of a pair (i_r, i_s) with $\Delta_{rs} = (r - s)(i_r - i_s) < 0$, (there is a conflict between the order and the value of the order statistics). N_m can be written as:

$$N_m = \sum_{r=1}^{m} \sum_{s=1}^{r-1} I(i_r < i_s), \qquad \ldots\ldots(5)$$

where $\sum_{s=1}^{0} I(i_r < i_s) = 0$, and $I(.)$ is the indicator function:

$$I(i_r < i_s) = \begin{cases} 1 & \text{if } i_r < i_s \\ 0 & \text{if } i_r \geq i_s \end{cases}$$

Clearly, the possible values of N_m are: $0,1,2,3, \dots, \frac{m(m-1)}{2}$.

The largest value of N_m, occur when,

$$N_m = \sum_{r=1}^{m} (r-1) = \frac{m(m-1)}{2}.$$

Reject H_0 if $N_m > c$, where c is obtained using $P_{H_0}(N_m > c) \leq \alpha$, where α is the significant level.

b. S_m : It is the sum of square of $(i_r - r)$, $r = 1,2, \dots m$, i.e.,

$$S_m = \sum_{r=1}^{m} (i_r - r)^2. \qquad \dots\dots (6)$$

It can be verified that the possible values of S_m are the even numbers: $0,2,4,6 \dots, \frac{1}{3}m(m^2-1)$.

The values of S_m, are even numbers since:

$$S_m = \sum_{r=1}^{m} (i_r - r)^2 = \sum_{r=1}^{m} i_r^2 + \sum_{r=1}^{m} r^2 - 2\sum_{r=1}^{m} r * i_r$$

$$= 2\sum_{r=1}^{m} r^2 - 2\sum_{r=1}^{m} r * i_r = 2(\sum_{r=1}^{m} r^2 - \sum_{r=1}^{m} r * i_r)$$

The largest value can be obtained as:

$$S_m = 2\sum_{r=1}^{m} r^2 - 2\sum_{r=1}^{m} r * i_r$$

$$= \frac{1}{3}m(m+1)(2m+1) - 2\sum_{r=1}^{m} r(m-(r-1))$$

$$= \frac{1}{3}m(m+1)(2m+1) - \frac{1}{3}m(m+1)(m+2)$$

$$= \frac{1}{3}m(m^2-1).$$

Reject H_0 if $S_m > c$, where c is obtained using $P_{H_0}(S_m > c) \le \alpha$.

c. A_m : It is the sum of the absolute value of the difference between i_r and r where $r = 1,2,\dots,m$, i.e.,

$$A_m = \sum_{r=1}^{m} |i_r - r|. \qquad \dots\dots(7)$$

The values of A_m are the set of the even numbers: $0,2,4,\dots,\left[\frac{m^2}{2}\right]$.

The values of A_m are the set of the even numbers since:

$$\sum_{r=1}^{m} |i_r - r| = 2\sum_{r=1}^{m} \max(i_r, r) - \sum_{r=1}^{m} (i_r + r)$$

$$= 2\sum_{r=1}^{m} \max(i_r, r) - \sum_{r=1}^{m} 2r = 2\left(\sum_{r=1}^{m} \max(i_r, r) - \sum_{r=1}^{m} r\right).$$

The largest value can be obtained as:

$$\sum_{r=1}^{m} |i_r - r| = \sum_{r=1}^{m} |m + 1 - 2r|$$

If r is odd, then $2r < m + 1$, so $r < \frac{m+1}{2}$

$$\sum_{r=1}^{m} |m + 1 - 2r| = \sum_{r=1}^{\frac{m+1}{2}-1} (m + 1 - 2r) + \sum_{r=\frac{m+1}{2}}^{m} (2r - m - 1).$$

$$= \frac{1}{2}(m^2 - 1)$$

If r is even, then $2r < m + 1$, so $r < \frac{m+1}{2}$, $r \leq \frac{m+1}{2} - \frac{1}{2}$, to get $r = \frac{m}{2}$.

$$\sum_{r=1}^{m} |m + 1 - 2r| = \sum_{r=1}^{m/2} (m - 1 - 2r) + \sum_{r=\frac{m}{2}+1}^{m} (2r - m - 1)$$

$$= \frac{1}{2}m^2.$$

So the largest value of $A_m = \left\{ \begin{array}{ll} \frac{1}{2}(m^2 - 1) & \text{if } m \text{ is odd} \\ \frac{1}{2}m^2 & \text{if } m \text{ is even} \end{array} \right\} = \left[\frac{1}{2}m^2\right].$

Reject H_0 if $A_m > c$, where c is obtained using $P_{H_0}(A_m > c) \leq \alpha$.

We can use the above test statistics to identify the rejection region for testing the hypothesis of perfect ranking, for specific α. Also, the tests can be compared via their powers.

Example: Suppose we have the following vectors: (i_1, i_2, i_3) =(1,2,3),(2,1,3), (3,2,1), (3,1,2); where $m = 3$; the values of the three tests are given in the following table.

(1,2,3)	(2,1,3)
$N_3 = I(3<2)+I(2<1)+I(3<1) = 0$	$N_3 = I(1<2)+I(3<2)+I(3<1) = 1$
$S_3 = (1-1)^2+(2-2)^2+(3-3)^2 = 0$	$S_3 = (1-2)^2+(2-1)^2+(3-3)^2 = 2$
$A_3 = \|1-1\|+\|2-2\|+\|3-3\| = 0$	$A_3 = \|1-2\|+\|2-1\|+\|3-3\| = 2$
(3,1,2)	(3,2,1)
$N_3 = I(1<3)+I(2<3)+I(2<1) = 2$	$N_3 = I(1<2)+I(1<3)+I(2<3) = 3$
$S_3 = (1-3)^2+(2-1)^2+(3-2)^2 = 6$	$S_3 = (1-3)^2+(2-2)^2+(3-1)^2 = 8$
$A_3 = \|1-3\|+\|2-1\|+\|3-2\| = 4$	$A_3 = \|1-3\|+\|2-2\|+\|3-1\| = 4$

2.6 Tables

For specific values of m, $m = 2,3,4,5$; the probability under H_0 is calculated and listed in Table (2-1). All values in this table are obtained using Formula (1). From Table (2-1), it can be noticed that the probability of any vector has larger value when the vector has the correct order and has smaller value when the vector of the correct order is reversed. For example, for $m = 3$:

$\pi_0(1,2,3) = \frac{1}{3}$, while $\pi_0(3,2,1) = \frac{1}{15}$.

Note that, as the set size increases, the probability of ranking decreases.

Table (2-1): Values of $\pi_0(i_1, i_2, \dots i_m)$ for $m = 2,3,4,5$ under H_0

m	$(i_1, i_2, \dots i_m)$	$\pi_0(i_1, i_2, \dots i_m)$	m	$(i_1, i_2, \dots i_m)$	$\pi_0(i_1, i_2, \dots i_m)$
2	(1,2)	2/3	5	(2,5,1,4,3)	1/168
	(2,1)	1/3		(2,5,3,4,1)	1/245
				(2,5,3,1,4)	2/385
3	(1,2,3)	1/3		(2,5,4,1,3)	1/231
	(1,3,2)	1/4		(2,5,4,3,1)	2/539
	(2,1,3)	1/6		(3,1,2,4,5)	1/90
	(2,3,1)	1/10		(3,1,2,5,4)	1/99
	(3,1,2)	1/12		(3,1,5,4,2)	2/351
	(3,2,1)	1/15		(3,1,5,2,4)	2/297
				(3,1,4,5,2)	1/156
4	(1,2,3,4)	2/15		(3,1,4,2,5)	1/120
	(1,3,2,4)	1/10		(3,2,1,4,5)	2/225
	(1,2,4,3)	4/35		(3,2,1,5,4)	4/495

	(1,4,2,3)	12/175		(3,2,5,4,1)	2/525
	(1,4,3,2)	3/50		(3,2,5,1,4)	4/825
	(1,3,4,2)	3/40		(3,2,4,5,1)	4/945
	(2,1,3,4)	1/15		(3,2,4,1,5)	4/675
	(2,3,4,1)	2/75		(3,4,1,2,5)	1/210
	(2,4,3,1)	1/45		(3,4,1,5,2)	1/273
	(2,4,1,3)	1/35		(3,4,2,5,1)	4/1323
	(2,1,4,3)	2/35		(3,4,2,1,5)	4/945
	(2,3,1,4)	1/25		(3,4,5,1,2)	2/819
	(3,2,1,4)	2/75		(3,4,5,2,1)	1/441
	(3,1,2,4)	1/30		(3,5,4,1,2)	1/468
	(3,2,4,1)	4/225		(3,5,4,2,1)	1/504
	(3,4,1,2)	1/70		(3,5,2,4,1)	1/420
	(3,4,2,1)	4/315		(3,5,2,1,4)	1/330
	(3,1,4,2)	1/40		(3,5,1,2,4)	1/297
	(4,1,2,3)	3/175		(3,5,1,4,2)	1/351
	(4,2,3,1)	1/90		(4,1,2,3,5)	1/175
	(4,2,1,3)	1/70		(4,1,2,5,3)	1/210
	(4,1,3,2)	3/200		(4,1,5,2,3)	1/300
	(4,3,1,2)	3/280		(4,1,5,3,2)	1/325
	(4,3,2,1)	1/105		(4,1,3,5,2)	1/260
				(4,1,3,2,5)	1/200
5	(1,2,3,4,5)	2/45		(4,2,1,3,5)	1/210
	(1,2,3,5,4)	4/99		(4,2,1,5,3)	1/252
	(1,2,4,5,3)	2/63		(4,2,5,1,3)	1/396
	(1,2,4,3,5)	4/105		(4,2,5,3,1)	1/462
	(1,2,5,4,3)	1/36		(4,2,3,5,1)	1/378
	(1,2,5,3,4)	1/33		(4,2,3,1,5)	1/270
	(1,3,2,4,5)	1/30		(4,3,1,2,5)	1/280
	(1,3,4,5,2)	1/52		(4,3,1,5,2)	1/364
	(1,3,5,4,2)	2/117		(4,3,2,1,5)	1/315
	(1,3,2,5,4)	1/33		(4,3,2,5,1)	1/441
	(1,3,4,2,5)	1/40		(4,3,5,1,2)	1/546
	(1,3,5,2,4)	2/99		(4,3,5,2,1)	1/588
	(1,4,5,3,2)	4/325		(4,5,1,2,3)	1/540
	(1,4,5,2,3)	1/75		(4,5,1,3,2)	1/585
	(1,4,3,5,2)	1/65		(4,5,2,1,3)	1/594
	(1,4,2,5,3)	2/105		(4,5,2,3,1)	1/693
	(1,4,2,3,5)	4/175		(4,5,3,2,1)	1/756
	(1,4,3,2,5)	1/50		(4,5,3,1,2)	1/702
	(1,5,4,2,3)	1/90		(5,1,2,3,4)	1/330
	(1,5,4,3,2)	2/195		(5,1,2,4,3)	1/360

	(1,5,3,4,2)	4/351		(5,1,3,2,4)	4/1485
	(1,5,3,2,4)	4/297		(5,1,3,4,2)	4/1755
	(1,5,2,4,3)	1/72		(5,1,4,2,3)	1/450
	(1,5,2,3,4)	1/66		(5,1,4,3,2)	2/975
	(2,1,3,4,5)	1/45		(5,2,1,3,4)	1/385
	(2,1,3,5,4)	2/99		(5,2,1,4,3)	1/420
	(2,1,4,3,5)	2/105		(5,2,3,4,1)	2/1225
	(2,1,4,5,3)	1/63		(5,2,3,1,4)	4/1925
	(2,1,5,4,3)	1/72		(5,2,4,1,3)	2/1155
	(2,1,5,3,4)	1/66		(5,2,4,3,1)	4/2695
	(2,3,1,4,5)	1/75		(5,3,1,2,4)	1/495
	(2,3,1,5,4)	2/165		(5,3,1,4,2)	1/585
	(2,3,4,5,1)	2/315		(5,3,2,1,4)	1/550
	(2,3,4,1,5)	2/225		(5,3,2,4,1)	1/700
	(2,3,5,1,4)	2/275		(5,3,4,1,2)	1/780
	(2,3,5,4,1)	1/175		(5,3,4,2,1)	1/840
	(2,4,3,1,5)	1/135		(5,4,1,2,3)	1/675
	(2,4,3,5,1)	1/189		(5,4,1,3,2)	4/2925
	(2,4,1,5,3)	1/126		(5,4,2,1,3)	2/1485
	(2,4,1,3,5)	1/105		(5,4,2,3,1)	4/3465
	(2,4,5,3,1)	1/231		(5,4,3,1,2)	2/1755
	(2,4,5,1,3)	1/198		(5,4,3,2,1)	1/945
	(2,5,1,3,4)	1/154			

The probability under specific H_1 are given in Table (2-2).

Table (2-2): Specific H_1.

Case	$m=2$	$m=3$	$m=4$	$m=5$
(1)	$H_1: a_{11} = 1, a_{12} = a_{22} = \frac{1}{2}$	$H_1: a_{11} = 1,\ a_{12} = a_{22} = \frac{1}{2}, a_{13} = a_{23} = a_{33} = \frac{1}{3}$	$H_1: a_{11} = 1, a_{12} = a_{22} = \frac{1}{2},\ a_{13} = a_{23} = a_{33} = \frac{1}{3}, a_{14} = a_{24} = a_{34} = a_{44} = \frac{1}{4}$	$H_1: a_{11} = 1,\ a_{12} = a_{22} = \frac{1}{2}, a_{13} = a_{23} = a_{33} = \frac{1}{3}, a_{14} = a_{24} = a_{34} = a_{44} = \frac{1}{4}, a_{15} = a_{25} = a_{35} = a_{45} = a_{55} = \frac{1}{5}$
(2)	$H_1: a_{11} = 1, a_{12} = \frac{2}{3}, a_{22} = \frac{1}{3}$	$H_1: a_{11} = 1, a_{12} = \frac{4}{5}, a_{22} = \frac{1}{5},\ a_{13} = \frac{8}{10}, a_{23} = \frac{1}{10}, a_{33} = \frac{1}{10}$	$H_1: a_{11} = 1, a_{12} = \frac{4}{5}, a_{22} = \frac{1}{5}, a_{13} = \frac{8}{10}, a_{23} = \frac{1}{10}, a_{33} = \frac{1}{10}, a_{14} = \frac{7}{12}, a_{24} = \frac{3}{12}, a_{34} = \frac{1}{12}, a_{44} = \frac{1}{12}$	$H_1: a_{11} = 1, a_{12} = \frac{4}{5}, a_{22} = \frac{1}{5}, a_{13} = \frac{9}{10}, a_{23} = 0\ a_{33} = \frac{1}{10}, a_{14} = \frac{7}{12}, a_{24} = \frac{3}{12}, a_{34} = 0, a_{44} = \frac{2}{12}, a_{15} = \frac{15}{24}, a_{25} = \frac{5}{24}, a_{35} = 0, a_{45} = \frac{1}{24}, a_{55} = \frac{3}{24}$
(3)	$H_1: a_{11} = 1, a_{12} = \frac{1}{10}, a_{22} = \frac{9}{10}$	$H_1: a_{11} = 1, a_{12} = \frac{1}{5}, a_{22} = \frac{4}{5},\ a_{13} = \frac{1}{10}, a_{23} = 0,\ a_{33} = \frac{9}{10}$	$H_1: a_{11} = 1, a_{12} = \frac{1}{5}, a_{22} = \frac{4}{5}, a_{13} = \frac{1}{10}, a_{23} = 0, a_{33} = \frac{9}{10}, a_{14} = \frac{1}{12}, a_{24} = 0, a_{34} = \frac{1}{12}, a_{44} = \frac{10}{22}$	$H_1: a_{11} = 1, a_{12} = \frac{1}{5}, a_{22} = \frac{4}{5}, a_{13} = \frac{1}{10}, a_{23} = 0, a_{33} = \frac{9}{10}, a_{14} = \frac{1}{12}, a_{24} = 0, a_{34} = \frac{1}{12}, a_{44} = \frac{10}{12}, a_{15} = \frac{3}{24}, a_{25} = \frac{1}{24}, a_{35} = 0, a_{45} = \frac{5}{24}, a_{55} = \frac{15}{24}$

For the above three cases and from the result in Table (2-3) it can be seen that:

Case (1): This case can be described for any value of m as follows:

$H_1: a_{ki} = \frac{1}{m}\ \forall k, k = 1,2,..,i.\ and\ i = 1,2,\ldots,m.$ (Ranking is as good as random; i.e.; MERSS is a SRS). In this case, for each value of m, the value of $\pi_1(i_1, i_2,..,i_m) = 1/m!$.

Case (2): The values of probability are increasing when the order of the vector getting worse.

For example, take $m = 4$,

$$\pi_1(1,2,3,4) = 0.019650794 \text{ and } \pi_1(4,3,2,1) = 0.082349206.$$

Case (3): The values of probability are decreasing when the order of the vector getting worse.

For example, take $m = 5$,

$\pi_1(1,2,3,4,5) = 0.026285858$ and $\pi_1(1,4,5,3,2) = 0.009253633$.

So, we can see that case (3) is very similar to H_0.

Table (2-3): Values of $\pi_1(i_1, i_2, \ldots i_m)$ for $m = 2,3,4,5$; under the specific H_1.

m	Case (1)	Case (2)	Case (3)
2			
(1,2)	0.5	0.444444	0.633333
(2,1)	0.5	0.555556	0.366667
3			
(1,2,3)	0.166667	0.092	0.292
(1,3,2)	0.166667	0.119	0.207
(2,1,3)	0.166667	0.114	0.201
(2,3,1)	0.166667	0.212	0.117
(3,1,2)	0.166667	0.189	0.101
(3,2,1)	0.166667	0.274	0.082
4			
(1,2,3,4)	0.041666667	0.019650794	0.106732
(1,3,2,4)	0.041666667	0.023976190	0.077286
(1,2,4,3)	0.041666667	0.018952381	0.094825

(1,4,2,3)	0.041666667	0.022658730	0.059738
(1,4,3,2)	0.041666667	0.028357143	0.049071
(1,3,4,2)	0.041666667	0.028880952	0.056512
(2,1,3,4)	0.041666667	0.022492063	0.074948
(2,3,4,1)	0.041666667	0.056976190	0.029286
(2,4,3,1)	0.041666667	0.055785714	0.025679
(2,4,1,3)	0.041666667	0.031896825	0.034845
(2,1,4,3)	0.041666667	0.021698413	0.066365
(2,3,1,4)	0.041666667	0.033880952	0.044726
(3,2,1,4)	0.041666667	0.040817460	0.033325
(3,1,2,4)	0.041666667	0.032976190	0.041000
(3,2,4,1)	0.041666667	0.067563492	0.020984
(3,4,1,2)	0.041666667	0.058857143	0.016095
(3,4,2,1)	0.041666667	0.083269841	0.014349
(3,1,4,2)	0.041666667	0.038992063	0.028806
(4,1,2,3)	0.041666667	0.030738095	0.030706
(4,2,3,1)	0.041666667	0.065357143	0.017310
(4,2,1,3)	0.041666667	0.037912698	0.024841
(4,1,3,2)	0.041666667	0.037785714	0.024131
(4,3,1,2)	0.041666667	0.058174603	0.015099
(4,3,2,1)	0.041666667	0.082349206	0.013341
5			
(1,2,3,4,5)	0.00833333	0.00320347	0.026285858
(1,2,3,5,4)	0.00833333	0.003231197	0.026204881
(1,2,4,5,3)	0.00833333	0.002295005	0.021632466
(1,2,4,3,5)	0.00833333	0.002865829	0.023414022
(1,2,5,4,3)	0.00833333	0.002588408	0.021788312
(1,2,5,3,4)	0.00833333	0.003308881	0.023544042
(1,3,2,4,5)	0.00833333	0.003974638	0.019390626
(1,3,4,5,2)	0.00833333	0.004630968	0.011557867
(1,3,5,4,2)	0.00833333	0.005316010	0.011600146
(1,3,2,5,4)	0.00833333	0.003959051	0.019333724
(1,3,4,2,5)	0.00833333	0.004631870	0.014216374
(1,3,5,2,4)	0.00833333	0.005258283	0.014278650
(1,4,5,3,2)	0.00833333	0.004718449	0.009253633
(1,4,5,2,3)	0.00833333	0.003526785	0.010557723
(1,4,3,5,2)	0.00833333	0.004454564	0.010132467
(1,4,2,5,3)	0.00833333	0.002650453	0.014046394
(1,4,2,3,5)	0.00833333	0.003349322	0.015272380
(1,4,3,2,5)	0.00833333	0.004415002	0.012531153
(1,5,4,2,3)	0.00833333	0.004031068	0.011411940
(1,5,4,3,2)	0.00833333	0.005351570	0.009953024
(1,5,3,4,2)	0.00833333	0.005831696	0.011013072
(1,5,3,2,4)	0.00833333	0.005797180	0.013756383
(1,5,2,4,3)	0.00833333	0.003411121	0.015648623

(1,5,2,3,4)	0.00833333	0.004381709	0.017059778
(2,1,3,4,5)	0.00833333	0.003541612	0.018730130
(2,1,3,5,4)	0.00833333	0.003553557	0.018673181
(2,1,4,3,5)	0.00833333	0.003160047	0.016634007
(2,1,4,5,3)	0.00833333	0.002515719	0.015343958
(2,1,5,4,3)	0.00833333	0.002824104	0.015484607
(2,1,5,3,4)	0.00833333	0.003611964	0.01676303
(2,3,1,4,5)	0.00833333	0.005148867	0.011420693
(2,3,1,5,4)	0.00833333	0.005061629	0.011388703
(2,3,4,5,1)	0.00833333	0.009709891	0.005666332
(2,3,4,1,5)	0.00833333	0.007458437	0.007492534
(2,3,5,1,4)	0.00833333	0.008207387	0.007517810
(2,3,5,4,1)	0.00833333	0.011148379	0.005676097
(2,4,3,1,5)	0.00833333	0.007040898	0.006651732
(2,4,3,5,1)	0.00833333	0.009287787	0.004999419
(2,4,1,5,3)	0.00833333	0.003313528	0.008314118
(2,4,1,3,5)	0.00833333	0.004246363	0.009056409
(2,4,5,3,1)	0.00833333	0.009678539	0.004584513
(2,4,5,1,3)	0.00833333	0.005275142	0.005624962
(2,5,1,3,4)	0.00833333	0.005476674	0.010024726
(2,5,1,4,3)	0.00833333	0.004259673	0.009174128
(2,5,3,4,1)	0.00833333	0.012276256	0.005310120
(2,5,3,1,4)	0.00833333	0.009080229	0.007158654
(2,5,4,1,3)	0.00833333	0.006099129	0.005974857
(2,5,4,3,1)	0.00833333	0.011063869	0.004835404
(3,1,2,4,5)	0.00833333	0.005349650	0.010940988
(3,1,2,5,4)	0.00833333	0.005253612	0.010912249
(3,1,5,4,2)	0.00833333	0.006682779	0.006223989
(3,1,5,2,4)	0.00833333	0.006744841	0.007825189
(3,1,4,5,2)	0.00833333	0.005891436	0.006164857
(3,1,4,2,5)	0.00833333	0.006109457	0.007726087
(3,2,1,4,5)	0.00833333	0.006277394	0.008908369
(3,2,1,5,4)	0.00833333	0.006118279	0.008885816
(3,2,5,4,1)	0.00833333	0.012915053	0.004197991
(3,2,5,1,4)	0.00833333	0.009671464	0.005673794
(3,2,4,5,1)	0.00833333	0.011345859	0.004178135
(3,2,4,1,5)	0.00833333	0.008949171	0.005625810
(3,4,1,2,5)	0.00833333	0.008599111	0.004443564
(3,4,1,5,2)	0.00833333	0.007908895	0.003500512
(3,4,2,5,1)	0.00833333	0.013419660	0.002887158
(3,4,2,1,5)	0.00833333	0.010887556	0.003943745
(3,4,5,1,2)	0.00833333	0.013079754	0.002433170
(3,4,5,2,1)	0.00833333	0.018330426	0.002253696
(3,5,4,1,2)	0.00833333	0.015257102	0.002567599
(3,5,4,2,1)	0.00833333	0.021316184	0.002371455

(3,5,2,4,1)	0.00833333	0.017730644	0.003105391
(3,5,2,1,4)	0.00833333	0.013720366	0.004330951
(3,5,1,2,4)	0.00833333	0.010890154	0.004951699
(3,5,1,4,2)	0.00833333	0.010368851	0.003833657
(4,1,2,3,5)	0.00833333	0.004281819	0.008742411
(4,1,2,5,3)	0.00833333	0.003338950	0.007942008
(4,1,5,2,3)	0.00833333	0.004341026	0.005699672
(4,1,5,3,2)	0.00833333	0.005723093	0.004881662
(4,1,3,5,2)	0.00833333	0.005473572	0.005382711
(4,1,3,2,5)	0.00833333	0.005593663	0.006880486
(4,2,1,3,5)	0.00833333	0.004936462	0.007069423
(4,2,1,5,3)	0.00833333	0.003820243	0.006421428
(4,2,5,1,3)	0.00833333	0.005987746	0.004097009
(4,2,5,3,1)	0.00833333	0.010852947	0.003244788
(4,2,3,5,1)	0.00833333	0.010504331	0.003562329
(4,2,3,1,5)	0.00833333	0.008128192	0.004916648
(4,3,1,2,5)	0.00833333	0.008302197	0.004355148
(4,3,1,5,2)	0.00833333	0.007667732	0.003383817
(4,3,2,1,5)	0.00833333	0.010517089	0.003831216
(4,3,2,5,1)	0.00833333	0.013025700	0.002754632
(4,3,5,1,2)	0.00833333	0.012731279	0.002299434
(4,3,5,2,1)	0.00833333	0.017855602	0.002120205
(4,5,1,2,3)	0.00833333	0.006781599	0.003668309
(4,5,1,3,2)	0.00833333	0.008655014	0.003078081
(4,5,2,1,3)	0.00833333	0.008325822	0.003193179
(4,5,2,3,1)	0.00833333	0.014669278	0.002459524
(4,5,3,2,1)	0.00833333	0.019845247	0.002043000
(4,5,3,1,2)	0.00833333	0.014159180	0.002229720
(5,1,2,3,4)	0.00833333	0.006239822	0.013635599
(5,1,2,4,3)	0.00833333	0.004844927	0.012341973
(5,1,3,2,4)	0.00833333	0.008161642	0.010526332
(5,1,3,4,2)	0.00833333	0.008002074	0.008124447
(5,1,4,2,3)	0.00833333	0.005626234	0.008449658
(5,1,4,3,2)	0.00833333	0.007302478	0.007201151
(5,2,1,3,4)	0.00833333	0.007173336	0.010757346
(5,2,1,4,3)	0.00833333	0.005571144	0.009728379
(5,2,3,4,1)	0.00833333	0.015522117	0.005140631
(5,2,3,1,4)	0.00833333	0.011759825	0.007240330
(5,2,4,1,3)	0.00833333	0.007856843	0.005874892
(5,2,4,3,1)	0.00833333	0.013928907	0.004607504
(5,3,1,2,4)	0.00833333	0.012050103	0.006369874
(5,3,1,4,2)	0.00833333	0.011394566	0.004856966
(5,3,2,1,4)	0.00833333	0.015171492	0.005526467
(5,3,2,4,1)	0.00833333	0.019438902	0.003876801
(5,3,4,1,2)	0.00833333	0.016681594	0.003150394

(5,3,4,2,1)	0.00833333	0.023263105	0.002893153
(5,4,1,2,3)	0.00833333	0.007702372	0.004653950
(5,4,1,3,2)	0.00833333	0.009749102	0.003908012
(5,4,2,1,3)	0.00833333	0.009450894	0.004060222
(5,4,2,3,1)	0.00833333	0.016472237	0.003126235
(5,4,3,1,2)	0.00833333	0.015877787	0.002831036
(5,4,3,2,1)	0.00833333	0.022200807	0.002592216

Null distribution and critical values of the test statistics

The following two tables; Table (2-4) and Table (2-5) contain the null distributions of the three tests for $m = 2,3,4,5$, and the critical values of N_m , S_m and A_m for nominal levels near 0.05 and 0.1 and the corresponding exact levels.

Table (2-4): Null distribution of the test statistics N_m, S_m and A_m for $m = 2,3,4,5$.

N_2	Probability	N_3	Probability	N_4	Probability	N_5	Probability
0	0.666666667	0	0.333333333	0	0.133333333	0	0.044444
1	0.333333333	1	0.416666667	1	0.280952381	1	0.134055
		2	0.183333333	2	0.274047619	2	0.203903
		3	0.066666667	3	0.184047619	3	0.216006
				4	0.083571429	4	0.171984
				5	0.03452381	5	0.114552
				6	0.00952381	6	0.069022
						7	0.03085
						8	0.009319
						9	0.004807
						10	0.001058
A_2	Probability	A_3	Probability	A_4	Probability	A_5	Probability
0	0.666666667	0	0.333333333	0	0.133333333	0	0.044444
2	0.333333333	2	0.416666667	2	0.280952381	2	0.134055
		4	0.25	4	0.360714286	4	0.260570
				6	0.177777778	6	0.279339
				8	0.047222222	8	0.183460
						10	0.066313
						12	0.031818
S_2	probability	S_3	Probability	S_4	Probability	S_5	Probability

0	0.666666667	0	0.333333333	0	0.133333333	0	0.0444444
2	0.333333333	2	0.416666667	2	0.280952381	2	0.1340555
		6	0.183333333	4	0.057142857	4	0.0695531
		8	0.066666667	6	0.216904762	6	0.134351
				8	0.086666667	8	0.109913
				10	0.053571429	10	0.079076
				12	0.043809524	12	0.048985
				14	0.069285714	14	0.097601
				16	0.026984127	16	0.043034
				18	0.021825397	18	0.066790
				20	0.009523810	20	0.031851
						22	0.038721
						24	0.019374
						26	0.028053
						28	0.009848
						30	0.012280
						32	0.012743
						34	0.009312
						36	0.005289
						38	0.003668
						40	0.001058

Table (2-5): Critical values (CV) of $N_m, S_m,$ and A_m for nominal levels near 0.05 and 0.1 and the corresponding exact levels.

m	N_m		A_m		S_m	
	CV	Exact level	CV	Exact level	CV	Exact level
3	3	0.06666667	******	******	8	0.066666667
4	5	0.04404762	8	0.047222222	16	0.058337624
	4	0.127619	*****	*****	14	0.12817889
5	7	0.046034	12	0.031818	28	0.05419800
	6	0.115056	10	0.098131	26	0.08225100

From Tables (2-4) and (2-5) it can be seen that when m has small values the nominal levels 0.05 and 0.1 cannot be achieved exactly so approximation values are given, for example:

$m = 5$, $\alpha = 0.05$ for S_m, we reject H_0 when $S_5 \geq 28$; with $P(S_5 \geq 28|H_0) = 0.054198$.

The next three tables are the distributions of the test statistics N_m, S_m *and* A_m for

$m = 2,3,4,5$ under H_1, for Case (1), Case (2) and Case (3), respectively.

Table (2-6): Distribution of test statistics N_m, S_m and A_m for $m = 2,3,4,5$ under H_1, for Case 1, Case 2, and Case 3, respectively.

Table (2-6, a): Case (1)

N_2	Probability	N_3	Probability	N_4	Probability	N_5	Probability
0	0.5	0	0.166667	0	0.041666667	0	0.00833333
1	0.5	1	0.333333	1	0.125	1	0.03333332
		2	0.333333	2	0.2083333	2	0.07499997
		3	0.166667	3	0.25	3	0.12499995
				4	0.2083333	4	0.16666666
				5	0.125	5	0.18333326
				6	0.041666667	6	0.18333326
						7	0.13333328
						8	0.04999998
						9	0.03333332
						10	0.00833333
A_2	Probability	A_3	Probability	A_4	Probability	A_5	Probability
0	0.5	0	0.166667	0	0.041666667	0	0.00833333
2	0.5	2	0.333333	2	0.125	2	0.03333332
		4	0.5	4	0.291666667	4	0.09999996
				6	0.375	6	0.19999992
				8	0.166666667	8	0.29166655
						10	0.19999992
						12	0.16666666
S_2	probability	S_3	Probability	S_4	Probability	S_5	Probability
0	0.5	0	0.166667	0	0.041666667	0	0.00833333
2	0.5	2	0.333333	2	0.125	2	0.03333332
		6	0.333333	4	0.041666667	4	0.02499999
		8	0.166667	6	0.166666667	6	0.04999998
				8	0.083333333	8	0.05833331
				10	0.083333333	10	0.04999998
				12	0.083333333	12	0.03333332
				14	0.166666667	14	0.08333333
				16	0.041666667	16	0.04999998
				18	0.125	18	0.08333333
				20	0.041666667	20	0.04999998
						22	0.08333333
						24	0.04999998

						26	0.08333333
						28	0.03333332
						30	0.04999998
						32	0.05833331
						34	0.04999998
						36	0.03333332
						38	0.02499999
						40	0.00833333

Table (2-6, b): Case (2)

N_2	Probability	N_3	Probability	N_4	Probability	N_5	Probability
0	0.444444	0	0.092	0	0.019650794	0	0.003203470
1	0.555556	1	0.233	1	0.065420634	1	0.013613276
		2	0.401	2	0.140095238	2	0.034756250
		3	0.274	3	0.227777778	3	0.068741217
				4	0.257904761	4	0.115489877
				5	0.206801588	5	0.161264265
				6	0.082349207	6	0.209574118
						7	0.197087599
						8	0.098610745
						9	0.075458376
						10	0.022200807
A_2	Probability	A_3	Probability	A_4	Probability	A_5	Probability
0	0.444444	0	0.092	0	0.019650794	0	0.003203470
2	0.555556	2	0.233	2	0.065420634	2	0.013613276
		4	0.675	4	0.209269844	4	0.048037054
				6	0.423007934	6	0.123861753
				8	0.282650793	8	0.272357044
						10	0.246523018
						12	0.292404385
S_2	probability	S_3	Probability	S_4	Probability	S_5	Probability
0	0.444444	0	0.092	0	0.019650794	0	0.00320347
2	0.555556	2	0.233	2	0.065420634	2	0.013613276
		6	0.401	4	0.021698413	4	0.010672655
		8	0.274	6	0.118396824	6	0.024083595
				8	0.069174603	8	0.029723728
				10	0.070888889	10	0.027206939
				12	0.087714286	12	0.020752933
				14	0.199047619	14	0.055557438
				16	0.058857143	16	0.035040343

				18	0.206801588	18	0.066333317
				20	0.082349206	20	0.046372173
						22	0.080377147
						24	0.051696517
						26	0.099673619
						28	0.046563539
						30	0.067696353
						32	0.086524132
						34	0.091739591
						36	0.061387839
						38	0.059580589
						40	0.022200807

Table (2-6, c): Case (3)

N_2	Probability	N_3	Probability	N_4	Probability	N_5	Probability
0	0.633333	0	0.292	0	0.106732	0	0.026285858
1	0.366667	1	0.408	1	0.247059	1	0.087739659
		2	0.218	2	0.268341	2	0.151667855
		3	0.082	3	0.206039	3	0.187595904
				4	0.111730	4	0.183827422
				5	0.046758	5	0.153185037
				6	0.013341	6	0.119570089
						7	0.059703678
						8	0.016938802
						9	0.010893424
						10	0.002592216
A_2	Probability	A_3	Probability	A_4	Probability	A_5	Probability
0	0.633333	0	0.292	0	0.106732	0	0.026285858
2	0.366667	2	0.408	2	0.247059	2	0.087739659
		4	0.3	4	0.350737	4	0.194895689
				6	0.236588	6	0.262033283
				8	0.058884	8	0.256658865
						10	0.114453997
						12	0.057932599
S_2	probability	S_3	Probability	S_4	Probability	S_5	Probability
0	0.633333	0	0.292	0	0.106732	0	0.026285858
2	0.366667	2	0.408	2	0.247059	2	0.087739659
		6	0.218	4	0.066365	4	0.054640912
		8	0.082	6	0.201976	6	0.097026943
				8	0.082396	8	0.097635774
				10	0.063651	10	0.069477963

				12	0.059992	12	0.044852599		
				14	0.095635	14	0.093504377		
				16	0.016095	16	0.046650688		
				18	0.046758	18	0.072387525		
				20	0.013341	20	0.045627536		
						22	0.068256147		
						24	0.032497657		
						26	0.054459095		
						28	0.020317654		
						30	0.023183516		
						32	0.023186577		
						34	0.020944199		
						36	0.010670674		
						38	0.008062388		
						40	0.002592216		

Power comparison

The distribution of test statistics N_m, S_m and A_m for $m = 2,3,4,5$ under H_1 for the three cases was seen in Tables (2-6, a, b, c). So we can identify which of the tests is the most powerful. Suppose that $\alpha = 0.1, m = 5$, then the rejection region and the power for each of the three test, are given in Tables (2-7, a, b, c). It can be seen that for the three cases, the best test is N_5, then A_5, then S_5.

Table (2-7): Power comparison when, $\alpha = 0.1, m = 5$, H_1: Case 1, Case2 and Case 3, respectively.

Table (2-7, a): Case (1)

	N_5	A_5	S_5
Rejection region	{6,7,8,9,10}	{10,12}	{26,28,30,32,34,36,38,40}
Approximate power	0.40833317	0.36666658	0.34166656

Table (2-7, b): Case (2)

	N_5	A_5	S_5
Rejection region	{6,7,8,9,10}	{10,12}	{26,28,30,32,34,36,38,40}
Approximate power	0.602931645	0.538927403	0.535366469

Table (2-7, c): Case (3)

	N_5	A_5	S_5
Rejection region	{6,7,8,9,10}	{10,12}	{26,28,30,32,34,36,38,40}
Approximate power	0.209698209	0.172386596	0.163416319

2.7 Application: Trees Data

In this section, data of heights and diameter of 1083 trees will be used. These data was collected by Pordan (1968). We use this data set to apply our results that we obtained earlier in this chapter. Figure (2-1) is the scatter plot of the height (Y) versus diameter (X); the empirical CDF of diameter and the height and the normal probability plot of the data are in Figure (2-2) and Figure (2-3), respectively. The data are given in Appendix (1). The correlation coefficient between diameter and height is $\rho = 0.721$. The description of the data is given in the following table:

Table (2-8): Descriptive statistics of the trees data

Variables	Mean	SE Mean	Standard Deviation	Minimum	Maximum
Diameter	23.070	0.190	6.268	11.65	57.25
Height	21.656	0.0924	3.039	12.00	30.20

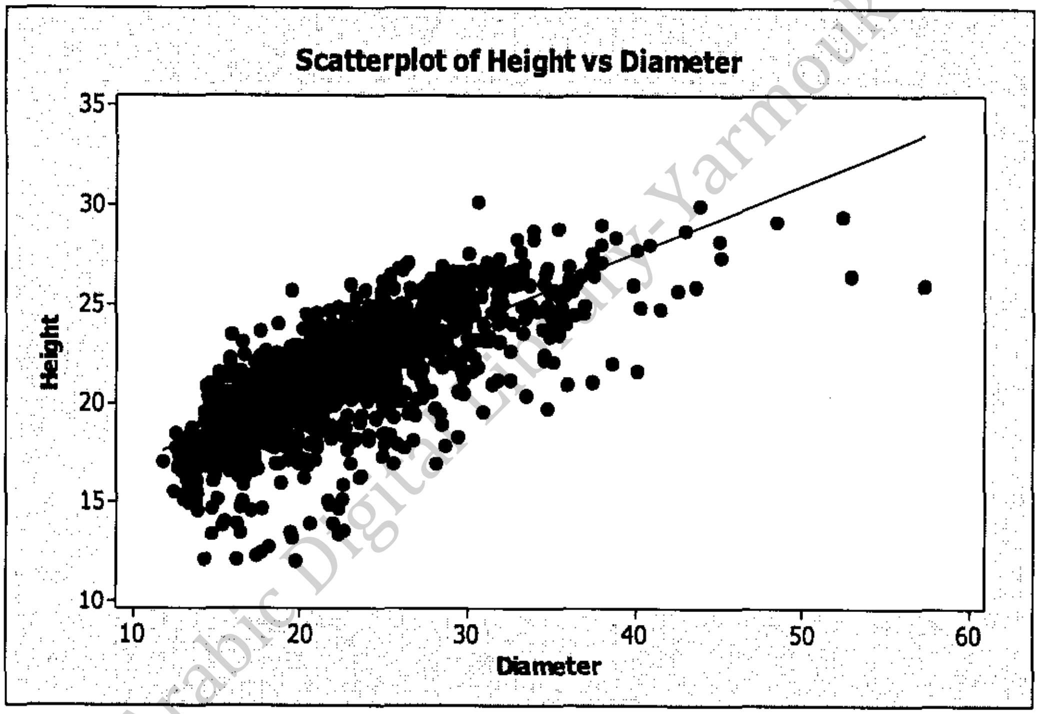

Figure (2-1)

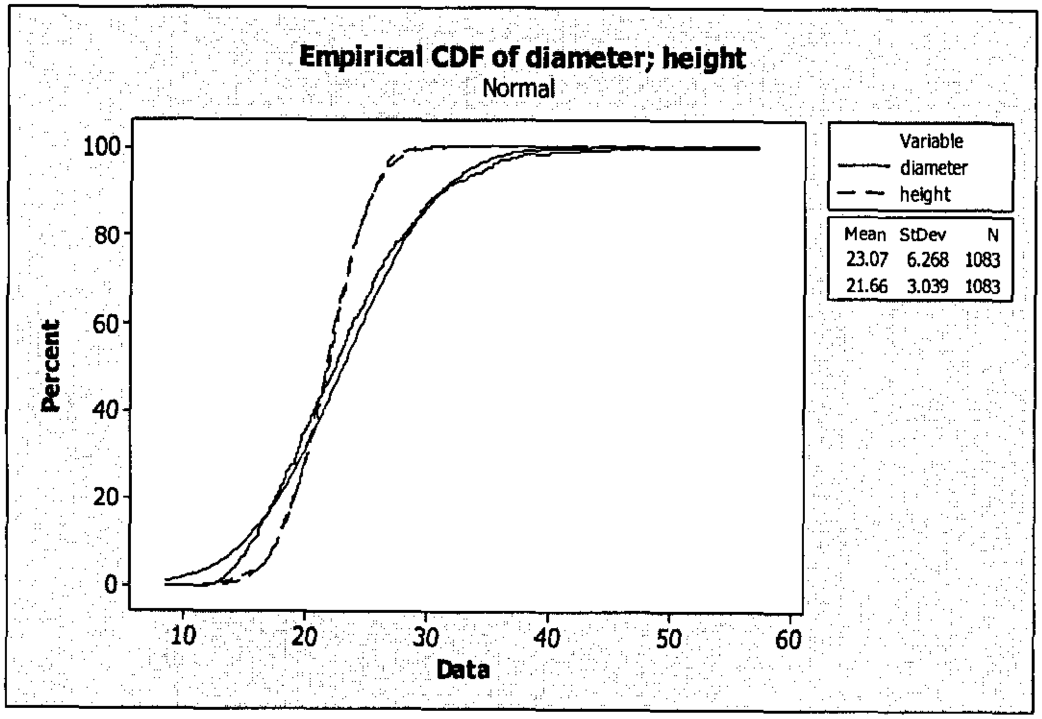

Figure (2-2)

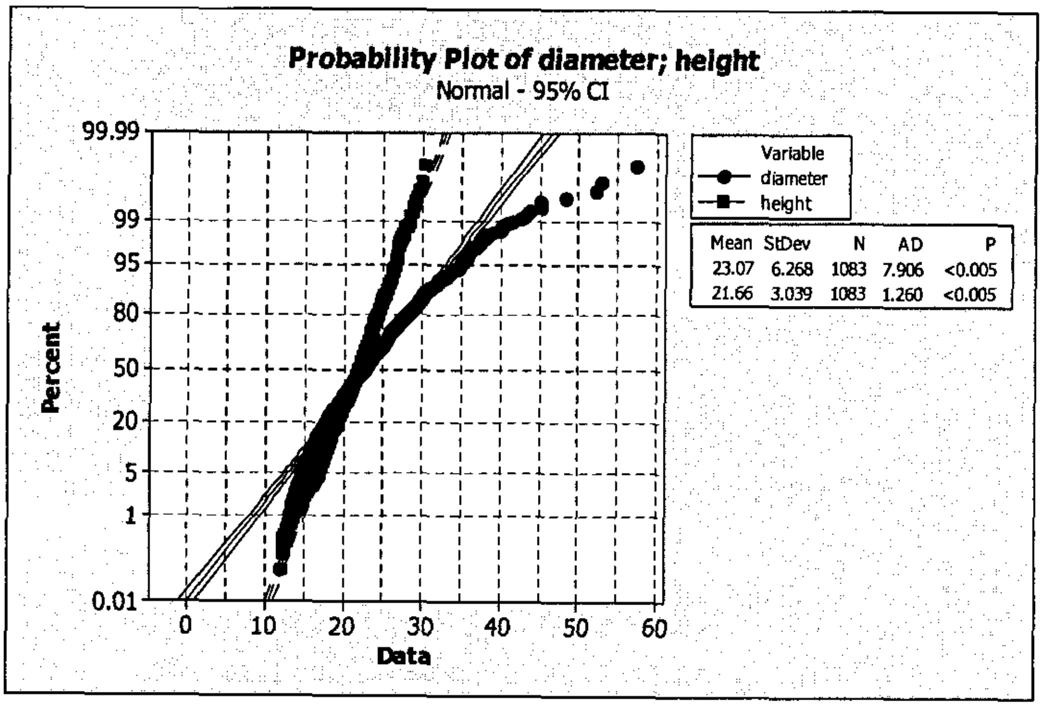

Figure (2-3)

Different MERSS samples are chosen from this data. Assume that (X, Y) is a bivariate data and suppose that the variable Y is difficult to measure or to order by judgment, but the variable X, which is correlated with Y, is easier to measure or to order by judgment. To choose a MERSS sample from this bivariate data we follow the following steps:

1) Choose SRS of size $1,2,\ldots\ldots,m$, respectively.
2) Identify by judgment the maximum of each set with respect to the variable X.
3) Measure accurately the selected judgment identified units for both variables.

This gives us a MERSS with concomitant variable.

In this example, X is representing the diameter of the trees, and Y represent the height of the trees. As mentioned in Section (2.2) our hypotheses are:

H_0: Ranking is perfect (no ranking error)

H_1: There is some ranking error (ranking is not perfect)

10000 MERSSs each of size $m = 5$ were chosen randomly as above, and the three tests for each sample are computed.

For example, take this sample to see how the value of each test can be obtained for any sample, and how the p-value can be computed:

Sample (X,Y)	Rearrange according to *X*
(25.55, 17.0)	(23.45, 23.0)
(23.45, 23.0)	(25.10, 20.8)
(26.65, 23.7)	(25.55, 17.0)
(25.10, 20.8)	(26.65, 23.7)
(37.25, 26.8)	(37.25, 26.8)

Then, determine the vector according to the value of the height and give them a number so that the smallest one is number (1) the second one number (2) and so on. Then according to this, our vector will be (3,2,1,4,5). $N_5 = 3$, $A_5 = 4$ and $S_5 = 8$.

The p-value (The probability is calculated under H_0) can be found as:

$P(N_5 \geq 3) = \sum_{i=3}^{10} P(N_5 \geq i) = 0.617598.$

The p- value for A_5 is $P(A_5 \geq 4) = 0.8125$

Finally, the p-value of S_5 is $P(S_5 \geq 8) = 0.617596.$

The above is repeated 10000 times and we get the following table:

Table (2-9): Summary of a simulation to find the average p-value, and the power of the three test statistics using MERSS from a population of 1083 trees.

Test	Average p-value	Number of rejection	Power of the test
N_5	0.507349015	856	0.0856
A_5	0.485116479	540	0.054
S_5	0.408224868	952	0.0952

The test with smaller average p-value, S_5, is the most sensitive test (the best one).

2.8. Concluding Remarks

In this chapter, the formula of $P(Y_{[i_1:i_1]} \leq Y_{[i_2:i_2]} \leq \cdots \leq Y_{[i_m:i_m]})$ under perfect ranking is derived; this formula is distribution free. Also, there is no-close form of $P(Y_{[i_1:i_1]} \leq Y_{[i_2:i_2]} \leq \cdots \leq Y_{[i_m:i_m]})$ under imperfect ranking, but at specific values of m and a's, this probability can be calculated easily.

According to these two probabilities, two hypotheses are constructed and tested. Also, we conjecture that the value of $H_0: P\left(Y_{[i_1:i_1]} \leq Y_{[i_2:i_2]} \leq \cdots \leq Y_{[i_m:i_m]}\right) = \frac{m!}{\prod_{k=1}^{m} \sum_{j=1}^{k} i_j}$, is equivalent to $H_0: a_{ki} = 1$ for $k = i$ and $a_{ki} = 0$ for $k \neq i$, where $i = 1,2, \dots m$. Several important properties are proved.

Also, some simple non-parametric tests were investigated. It was noticed that these test statistics are easy to use to check of the error in ranking. For the three test statistics, N_m, S_m, and A_m, the exact null distributions are obtained for $m = 2,3,4,5$. Also, under error in ranking, the exact power functions are computed for some values of m. All test statistics depend on the distance between $(i_1, i_2, \dots, i_m)$ and $(1,2,3, \dots, m)$. The smaller is the distance, the stronger is the evidence that H_0 is true and vise versa.

10000 different MERSSs each of size $m = 5$ from real data (trees data) are used, to find the values of the three tests and the corresponding p-value. So we can determine which of the three tests is the best one. According to the tables of results.

Chapter 3

Test for Perfect and Imperfect Ranking in Multi-Cycle MERSS

3.1. Introduction

In this chapter, we will discuss tests that deal with multi-cycle MERSS to test if the ranking is perfect or there is error in ranking. Multi-cycle MERSS is one way to increase the sample size by taking several cycles of MERSS for fixed set size m. In this way, the error in ranking is kept small. First, one-cycle MERSS of size m is chosen as given previously, and then this procedure is repeated r times to get a sample of size $n = mr$.

This chapter is arranged as follows; basic terminology are given in section 3.2. In section 3.3, the tests used in chapter (2) will be extended to multi- cycle MERSS. In section 3.4, chi-square test statistic will be introduced. Section 3.5, contains numerical applications, and concluding remarks are given in Section 3.6.

3.2. Basic Terminology

Suppose that $Y_{MERSS} = \{ Y_{[i:i]}{}^{(j)}, i = 1,2, \dots, m, j = 1,2, \dots, r\}$ is a multi-cycle MERSS. The elements of the sample can be displayed as follows:

$$Y_{[1:1]}{}^{(1)}, Y_{[2:2]}{}^{(1)}, \dots \dots \dots, Y_{[m:m]}{}^{(1)}$$

$$Y_{[1:1]}{}^{(2)}, Y_{[2:2]}{}^{(2)}, \dots \dots \dots, Y_{[m:m]}{}^{(2)}$$

$$\vdots$$

$$Y_{[1:1]}{}^{(r)}, Y_{[2:2]}{}^{(r)}, \dots \dots \dots, Y_{[m:m]}{}^{(r)}$$

Note that for $i = 1,2,..,m$, $\{Y_{[i:i]}{}^{(j)}, j = 1,2,..,r\}$ are independent identically distributed (iid) random variables. i.e. $Y_{[i:i]}{}^{(j)} \sim f_{[i:i]}(y^{(j)})$ (i.e. the columns are iid), so their joint density can be written as:

$$\left(Y_{[1:1]}{}^{(1)}, Y_{[1:1]}{}^{(2)}, \dots\dots, Y_{[1:1]}{}^{(r)}\right) \sim \prod_{j=1}^{r} f_{[1:1]}(y^{(j)}).$$

$$\left(Y_{[2:2]}{}^{(1)}, Y_{[2:2]}{}^{(2)}, \dots\dots, Y_{[2:2]}{}^{(r)}\right) \sim \prod_{j=1}^{r} f_{[2:2]}(y^{(j)}).$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\left(Y_{[m:m]}{}^{(1)}, Y_{[m:m]}{}^{(2)}, \dots\dots, Y_{[m:m]}{}^{(r)}\right) \sim \prod_{j=1}^{r} f_{[m:m]}(y^{(j)}).$$

3.3. Test Statistics for Multi-Cycle MERSS

In section 2.5, we investigated three test statistics: $N_m, A_m,$ and S_m for one cycle MERSS. Here, the same test statistics are extended for more than one cycle MERSS, and six test statistics are obtained from the original three test statistics. Since all cycles are independent of each other, we can use the null distribution of $N_m, A_m,$ and S_m that was computed in section (2.6) to find the null distribution of the six test statistics.

Let N_{mj} be the value of the test statistic N_m for the j^{th} cycle; A_{mj} be the value of the test statistic A_m for the j^{th} cycle, and S_{mj} be the value of the test statistic S_m for the j^{th} cycle. Three pairs of tests are given below. These tests were used to test for error in ranking in RSS by Li and Balakrishnan (2008). But instead of taking the maximum as they did, we take the minimum for the values of each test among cycles.

(1) $N_{m,r} = \sum_{j=1}^{r} N_{mj} \qquad N_{m,r}^{*} = \min(N_{m1}, \dots\dots, N_{mr}).$

(2) $A_{m,r} = \sum_{j=1}^{r} A_{mj}$ $\quad A^*_{m,r} = \min(A_{m1}, \dots\dots, A_{mr})$.

(3) $S_{m,r} = \sum_{j=1}^{r} S_{mi}$ $\quad S^*_{m,r} = \min\ (S_{m1}, \dots\dots, S_{mr})$.

It will be seen later in Section 3.5, that taking the minimum for each test increases the probability of type II error, i.e. $P(accept\ H_0|H_1\ is\ true)$, also take the maximum for each test increases the probability of type I error, i.e. $P(rejcet\ H_0|H_0\ is\ true)$. So, depending on which type of error is more important to be controlled, are chosen between minimum or maximum.

In general if T_j denotes the value of any of $N_m, A_m,$ or S_m for cycle j, $j = 1,2,..,r$. Then, the general form of above test statistics is as follows:

$$T = \sum_{j=1}^{r} T_j \qquad \text{or} \qquad T_{(r)} = \min(T_1, \dots\dots, T_r)$$

Since for $j = 1, \dots, r$, T_j's are independent and identically distributed, the distribution of T_j is the same distribution that we found in Section 2.6:

$$P(T = t) = \sum_{(t_1,t_2,\dots,t_r):\sum_{j=1}^{r} t_j = t} P(T_1 = t_1, T_2 = t_2, \dots\dots T_r = t_r).$$

$$= \sum_{(t_1,t_2,\dots,t_r):\sum_{j=1}^{r} t_j = t} \prod_{j=1}^{r} P(T_j = t_j).$$

For $T_{(r)} = \min(T_1, T_2, \cdots, T_r)$, the distribution function can be expressed as follows:

$$P(T_{(r)} = t) = P(\text{at least one of them} = \text{t, and other are} > t)$$

$$= P(T_{(r)} \geq t) - P(T_{(r)} > t)$$

$$= \left(P(T_{(1)} \geq t)\right)^r - \left(P(T_{(1)} > t)\right)^r.$$

It can be noticed that large value of any of the above test statistics supports H_1.

The smallest value of $N_{m,r}$ is $r * 0 = 0$ and the largest value is $r * \frac{m(m-1)}{2} = \frac{n(m-1)}{2}$. Similarly, the smallest value of $N^*_{m,r}$ is also 0 and the largest value is $\frac{m(m-1)}{2}$.

Also, the smallest value of $S_{m,r}$ is 0 and the largest value is $r * \frac{1}{3}m(m^2 - 1) = \frac{1}{3}n(m^2 - 1)$. Similarly, the smallest value of $S^*_{m,r}$ is 0 and the largest value is $\frac{1}{3}m(m^2 - 1)$. The smallest value of $A_{m,r}$ is 0 and the largest value is $r * \left[\frac{m^2}{2}\right]$. Finally, the smallest value of $A^*_{m,r}$ is 0 and the largest value is $\left[\frac{m^2}{2}\right]$.

Example: Assume that $r = 2,\ m = 3$, then $n = 6$. Based on Table (2-4), the null distributions of the above test statistics are given in Tables (3-1) and (3-3). Also, there critical values for $\alpha = 0.05$ and 0.1 are given in Tables (3-2) and (3-4).

Table (3-1): Null distribution of the tests $N_{3,2}, A_{3,2}$ and $S_{3,2}$.

$N_{3,2}$	Probability	$A_{3,2}$	Probability	$S_{3,2}$	Probability
0	0.111111111	0	0.111111111	0	0.111111111
1	0.277777778	2	0.277777778	2	0.277777778
2	0.295833333	4	0.340277778	4	0.173611111
3	0.197222222	6	0.208333333	6	0.122222222
4	0.089166667	8	0.0625	8	0.197222222
5	0.024444444			10	0.055555556
6	0.004444444			12	0.033611111
				14	0.024444444
				16	0.004444444

Table (3-2): Critical values of $N_{3,2}, A_{3,2}, S_{3,2}$ for nominal levels near 0.05 and 0.1 and corresponding exact levels.

n	$N_{3,2}$		$A_{3,2}$		$S_{3,2}$	
6	CV	Exact level	CV	Exact level	CV	Exact level
	5	0.028888888	8	0.0625	12	0.0625
	4	0.118055555	****	*****	10	0.118055556

Table (3-3): Null distribution of the tests $N^*_{3,2}, A^*_{3,2}, S^*_{3,2}$.

$N^*_{3,2}$	Probability	$A^*_{3,2}$	Probability	$S^*_{3,2}$	Probability
0	0.555555555	0	0.555555555	0	0.555555555
1	0.381944445	2	0.381944445	2	0.381944445
2	0.058055556	4	0.0625	6	0.058055556
3	0.004444444			8	0.004444444

Table (3-4): Critical values of $N^*_{3,2}, A^*_{3,2}, S^*_{3,2}$ for nominal levels near 0.05 and 0.1 and corresponding exact levels.

n	$N^*_{3,2}$		$A^*_{3,2}$		$S^*_{3,2}$	
6	CV	Exact level	CV	Exact level	CV	Exact level
	2	0.0625	4	0.0625	6	0.625
	*****	****	****	*****	****	****

<u>Power Comparison:</u>

Here, we want to investigate which of the tests is the most powerful test among the six tests for specific alternatives. For our example with $m = 3, r = 2$, it can be concluded from Table (3-5, a) that the most powerful test is $S_{3,2}$, then $N_{3,2}$ and at last $A_{3,2}$. Also, from Table (3-5, b) and Table (3-5, c) the most powerful tests are $A_{3,2}$ and $S_{3,2}$ with the same power, then $N_{3,2}$.

Also, it can be noticed from Table (3-6, a, b, c) that the three tests $N^*_{3,2}$,$A^*_{3,2}$, and $S^*_{3,2}$ give us the same power for each case of H_1.

Table (3-5): Power for $N_{3,2}$, $A_{3,2}$ and $S_{3,2}$ for $\alpha = 0.05, m = 3, r = 2, H_1$: Case 1, Case 2 and Case 3 respectively.

Table (3-5, a): Case (1)

	$N_{3,2}$	$A_{3,2}$	$S_{3,2}$
Rejection region	{5,6}	{8}	{12,14,16}
Approximate power	0.138889111	0.111110889	0.25

Table (3-5, b): Case (2)

	$N_{3,2}$	$A_{3,2}$	$S_{3,2}$
Rejection region	{5,6}	{8}	{12,14,16}
Approximate power	0.294824	0.455625	0.455625

Table (3-5, c): Case (3)

	$N_{3,2}$	$A_{3,2}$	$S_{3,2}$
Rejection region	{5,6}	{8}	{12,14,16}
Approximate power	0.042476	0.09	0.09

Table (3-6): Power for $N^*_{3,2}$, $A^*_{3,2}$ and $S^*_{3,2}$ for $\alpha = 0.05, m = 3, r = 2, H_1$: Case 1, Case 2 and Case 3 respectively.

Table (3-6, a): Case (1)

	$N^*_{3,2}$	$A^*_{3,2}$	$S^*_{3,2}$
Rejection region	{2,3}	{4}	{6,8}
Approximate power	0.25	0.25	0.25

Table (3-6, b): Case (2)

	$N^*_{3,2}$	$A^*_{3,2}$	$S^*_{3,2}$
Rejection region	{2,3}	{4}	{6,8}
Approximate power	0.455625	0.455625	0.455625

Table (3-6, c): Case (3)

	$N^*_{3,2}$	$A^*_{3,2}$	$S^*_{3,2}$
Rejection region	{2,3}	{4}	{6,8}
Approximate power	0.09	0.09	0.09

3.4. Chi-Square Test for Multi-Cycle MERSS

In this test, we compare observed values of an experiment with theoretical or expected values under the hypothesis H_0. Chi-square test is used to determine if there is a significant difference between the expected and the observed frequencies.

Let $H_0: p_1 = p_{10}, p_2 = p_{20}, \dots\dots, p_m = p_{m0}$ and H_1: H_0 is not true.

Let O_i be the observed frequency and E_i: be the expected frequency.

To apply the Chi-square test on multi-cycle MERSS we need to do the following:

- Identify our hypotheses as follows:

For any cycle, π_0 of any of the $m!$ permutation is $\frac{m!}{\prod_{k=1}^{m}\sum_{j=1}^{k} i_j}$, so our hypotheses are:

$H_0: P_{(i_1,i_2,\dots,i_m)} = \pi_0(i_1, i_2, i_3, \dots\dots, i_m) = \frac{m!}{\prod_{k=1}^{m}\sum_{j=1}^{k} i_j}$, versuse H_1: H_0 is not true

where $(i_1, i_2, \dots, i_m)$ is any of the $m!$ permutation of $(1,2,\dots,m)$

For example, if $m = 2$, then $p_{10} = \frac{2}{3}$, $p_{20} = \frac{1}{3}$.

- Find E_i as follows:

$$E_i = r * \pi_0(i_1, i_2, \ldots\ldots, i_m) = r * \frac{m!}{\prod_{k=1}^{m} \sum_{j=1}^{k} i_j}$$

The test statistic is $Q = \sum_{i=1}^{m!} \frac{(O_i - E_i)^2}{E_i}$.

Then, we reject H_0 for large value of Q. It is well known that as $r \to \infty$ the distribution of Q approaches the Chi-square distribution with $m! - 1$ degrees of freedom. H_0 is rejected if the observed value of Q is large.

Table (3-7): Details of the x^2- test.

Outcomes	O_i	E_i	${(O_i - E_i)^2}/{E_i}$
$(1,2,3,4,5,6, \ldots\ldots, m)$	O_1	rp_{10}	${(O_1 - E_1)^2}/{E_1}$
$(1,3,2,4,5,6 \ldots\ldots, m)$	O_2	rp_{20}	${(O_2 - E_2)^2}/{E_2}$
⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮
$(m, \ldots\ldots, 5,4,3,2,1)$	$O_{m!}$	$rp_{m!0}$	${(O_{m!} - E_{m!})^2}/{E_{m!}}$
Total	r	r	Q

$$Q = \sum_{i=1}^{m!} {(O_i - E_i)^2}/{E_i}$$

3.5. Numerical Applications

20,000 bivariate values are simulated from the bivariate normal distribution with $\mu_1 = \mu_2 = 0, \sigma_1 = \sigma_2 = 1$, and three different values of ρ, $\rho = 0.1, 0.5, 0.9$. This data is regarded as a population from which the MERSS samples are obtained.

Assume that $m = 3$ and $r = 10$. We find the distribution of the test statistic $N_{3,10}$ for different values of ρ. The approximate distribution of the test statistic $N_{3,10}$ for $\rho = 0.1, 0.5, 0.9$ is given in Table (3-8), which is obtained based on 10,000 iterations.

Table (3-8): The distribution of $N_{3,10}$ for $m = 3$ and $r = 10$, with 10,000 iteration.

Value of $N_{3,10}$	Probability for $\rho = 0.1$	Probability for $\rho = 0.5$	Probability for $\rho = 0.9$
0	0	0	0
1	0	0	0.0004
2	0	0.0002	0.0008
3	0	0.0003	0.0036
4	0.0001	0.0015	0.0122
5	0.0003	0.0052	0.0287
6	0.0023	0.0130	0.0560
7	0.0056	0.0295	0.0893
8	0.0125	0.0503	0.1182
9	0.0234	0.0771	0.1300
10	0.0474	0.1014	0.1453
11	0.0660	0.1215	0.1304
12	0.0991	0.1337	0.1052
13	0.1129	0.1313	0.0754
14	0.1234	0.1129	0.0464
15	0.1316	0.0836	0.0289
16	0.1177	0.0550	0.0163
17	0.0959	0.0394	0.0077
18	0.0690	0.0233	0.0036
19	0.0430	0.0112	0.0007
20	0.0252	0.0058	0.0006
21	0.0142	0.0023	0.0002
22	0.0062	0.0009	0.0001
23	0.0026	0.0005	0
24	0.0010	0.0001	0

25	0.0004	0	0
26	0.0001	0	0
27	0.0001	0	0
28	0	0	0
29	0	0	0
30	0	0	0

To find the rejection region for $N_{3,10}$ the same steps are done but for $\rho = 1$. We reject H_0 if $N_{3,10} > c$ where c can be obtained by solving $P_{H_0}(N_{3,10} > c) \leq \alpha = 0.05$.

After solving this equation, the rejection region is:

$C = \{15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30\}$ with exact significant level of $\alpha = 0.0551$.

The approximation power for $N_{3,10}$ is given in the following table.

Table (3-9): Approximation power for $N_{3,10}$ when $\rho = 0.1, 0.5, 0.9$ and $\alpha = 0.05$.

$N_{3,10}$	$\rho = 0.1$	$\rho = 0.5$	$\rho = 0.9$
Rejection region	$\{15,16,17,..,30\}$	$\{15,16,17,..,30\}$	$\{15,16,17,..,30\}$
Approximation Power	0.507	0.2221	0.0581

So, the chance of rejecting H_0 is getting small as ρ is getting large. We find the distribution of the test statistic $S^*_{3,10}$ for different values of ρ. The distribution of the test statistic $S^*_{3,10}$, for $\rho = 0.1, 0.5, 0.9$ is given in Table (3-10), which is obtained by using 10,000 iterations.

Table (3-10): *10,000* simulated distribution of $S^*_{3,10}$ for $m = 3$ and $r = 10$.

Value	Probability for $\rho = 0.1$	Probability for $\rho = 0.5$	Probability for $\rho = 0.9$
0	0.8634	0.9346	0.9856
2	0.1358	0.0654	0.0144
6	0.0008	0	0
8	0	0	0

We reject H_0 if $S^*_{3,10} > c$ where c can be obtained by solving $P_{H_0}\left(S^*_{3,10} > c\right) \leq 0.01$.

After solving the above inequality, the rejection region is $C = \{2,6,8\}$ with exact level of $\alpha = 0.01734153$.

Now, we want to find the power function for the different values of ρ. From the following table, the best result can be obtained when $\rho = 0.9$, with power equal to $P(reject\ H_0|H_1) = 0.0144$.

Table (3-11): Approximation power for $S^*_{3,10}$ when, $\rho = 0.1, 0.5, 0.9, \alpha = 0.01$.

$S^*_{3,10}$	$\rho = 0.1$	$\rho = 0.5$	$\rho = 0.9$
Rejection region	{2,6,8}	{2,6,8}	{2,6,8}
Approximation Power	0.1366	0.0654	0.0144

Also, we computed the Chi-square test for the three values of ρ. First for $m = 3$ and $r = 50$ to get total size $n = 150$. The degree of freedom is $m! - 1 = 5$, so the critical value is $x^2{}_{0.05,5} = 11.07$.

Table (3-12): Chi-square Test for $m = 3$, $r = 50$, $n = 150$ and $\rho = 0.1$.

Outcomes	O_i	P_i	E_i	$(O_i - E_i)^2/E_i$
(1,2,3)	12	1/3	16.6666667	1.30666668
(1,3,2)	10	1/4	12.5	0.5
(2,1,3)	3	1/6	8.33333333	3.41333333
(2,3,1)	8	1/10	5	1.8
(3,1,2)	8	1/12	4.16666667	3.52666666
(3,2,1)	9	1/15	3.33333333	9.63333335

Then,

$Q = \sum_{i=1}^{6} {(O_i - E_i)^2}/{E_i} = 20.18.$

Since $Q > 11.07$, we reject H_0 i.e. the ranking is not perfect.

Based on 5000 values of Q , the power of the test is $P(reject\ \ H_0|H_1) = 0.8718$.

For $\rho = 0.5$, the summary of the results is given in Table (3-13).

Table (3-13): Chi-square Test for $m = 3,\ r = 50,\ n = 150$ and $\rho = 0.5$.

Outcomes	O_i	P_i	E_i	${(O_i - E_i)^2}/{E_i}$
(1,2,3)	11	1/3	16.6666667	1.926666667
(1,3,2)	13	1/4	12.5	0.02
(2,1,3)	9	1/6	8.33333333	0.053333333
(2,3,1)	7	1/10	5	0.8
(3,1,2)	5	1/12	4.16666667	0.166666667
(3,2,1)	5	1/15	3.33333333	0.833333333

Then,

$Q = \sum_{i=1}^{6} {(O_i - E_i)^2}/{E_i} = 3.8$, $x^2{}_{0.05,5} = 11.07$, since $Q < 11.07$ we don't reject H_0 i.e. the ranking is perfect.

Based on 5000 values of Q, then, the power of the test will be $P(reject\ \ H_0|H_1) = 0.3276$.

The calculations for $\rho = 0.9$ is given in Table (3-14).

Table (3-14): Chi-square Test for $m = 3,\ r = 50,\ n = 150$ and $\rho = 0.9$.

Outcomes	O_i	P_i	E_i	$(O_i - E_i)^2 / E_i$
(1,2,3)	13	$1/3$	16.6666667	0.806666667
(1,3,2)	16	$1/4$	12.5	0.98
(2,1,3)	12	$1/6$	8.33333333	1.613333333
(2,3,1)	6	$1/10$	5	0.2
(3,1,2)	3	$1/12$	4.16666667	0.326666667
(3,2,1)	0	$1/15$	3.33333333	3.333333333

$$Q = \sum_{i=1}^{6} {(O_i - E_i)^2}/{E_i} = 7.26.$$

Compare with $x^2{}_{0.05,5} = 11.07$, since $Q < C$ then we don't reject H_0 i.e. the ranking is perfect. The approximation power based on simulation is $P(reject\ H_0|H_1) = 0.0524$.

3.6. Concluding Remarks

Multi-cycle MERSS is useful way to increase the sample size by taking several cycles of MERSS for fixed set size m. In this case, the error in ranking is kept small. The proposed tests are easy to apply on data, and their null distributions are easy to obtain.

The tests are illustrated using simulated data set from a bivariate normal distribution. It is noted large values of ρ, makes it difficult for any of the tests to reject H_0. This implies that if the correlation between X and Y is high, we may assume that the ranking is almost perfect.

Chapter 4

Conclusions and Suggestions for Further Research

4.1 General Concluding Remarks

In most of the work on RSS and MERSS, it is assumed that the ranking is perfect; an assumption which is most likely not true in practice. Before using the RSS or MERSS, the amount of ranking error should be evaluated. If it is minor then we may use the sample for inference assuming perfect ranking. If the ranking error is significant, then some modification should be done on the method of inference.

Recently, testing for imperfect ranking in RSS was discussed by Li and Balakrishnan (2008), and Vock and Balakrishnan (2011). In this thesis, our main concern was to test for ranking error in MERSS. We have investigated the suitability of some available simple non-parametric test statistics. All the tests are based on the distance between any permutations of $(1,2,..,m)$, $(i_1, i_2, ..., i_m)$, and $(1,2,..,m)$. The tests were investigated in the case of one-cycle MERSS and multi-cycle MERSS. For each test, we have found the null distribution. Also, we have computed the exact power functions under error in ranking. The computations were based on the formula obtained for $P(Y_{[i_1:i_1]} \leq Y_{[i_2:i_2]} \leq \cdots \leq Y_{[i_m:i_m]})$. The formula was derived under the general assumption of continuous distribution. The same test statistics were extended for multi-cycle MERSS. Also, we have used Chi-square test to test for perfect ranking in case of multi-cycle MERSS.

The tests were illustrated using real data for one cycle MERSS. For multi-cycle MERSS, some of the tests were illustrated using simulated data set from a bivariate normal distribution. It was noted that a value of $\rho = 0.9$ makes it difficult for any of the tests to reject H_0, the hypothesis of perfect ranking.

4.2. Suggested Future Work

The following are some suggested future works:

- Making inference about the underlying distribution taking into account, the test of perfect ranking. In view of the result of the test, the method of inference can be decided.
- Other tests can be used to test for perfect ranking. For example, Jonckheere-Terpstra- type test; this test compares pairs of measured values not just within a cycle but also across cycles. For more information on this test see Vock and Balakrishnan (2011).
- Testing for perfect ranking can be applied on specific bivariate distribution where one of the variables is considered as a concomitant variable. In this case, the hypothesis of perfect ranking may be rewritten in terms of the parameters and hence some parametric tests may be used. For example, for bivariate normal distribution, the hypothesis of perfect ranking can be rewritten in terms of ρ.
- Testing for error in ranking for other variations of RSS can be also considered.

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Appendix

Data of height and diameter of 1083 trees

Pordan (1968)

Diameter	Height	Diameter	Height	Diameter	Height	Diameter	Height	Diameter	Height
11.65	17.10	38.50	22.00	34.45	22.50	32.40	21.20	30.40	23.20
12.30	15.50	38.80	28.40	34.45	26.10	32.50	22.70	30.55	21.90
12.45	17.80	39.80	26.00	34.50	22.20	32.50	26.60	30.55	30.20
12.45	18.60	40.05	21.70	34.55	24.70	32.60	24.10	30.65	26.60
12.60	16.80	40.05	27.70	34.55	26.50	32.70	26.00	30.75	25.40
12.65	17.40	40.30	24.90	34.65	19.80	32.80	26.70	30.80	19.70
12.75	16.60	40.80	28.00	34.65	26.80	32.90	28.30	30.80	23.20
12.85	15.20	41.40	24.80	34.70	25.00	32.00	25.10	30.80	26.10
12.85	17.60	42.55	25.70	34.80	25.70	33.00	26.10	30.85	26.30
12.95	16.50	42.90	28.70	34.85	23.40	33.05	26.60	30.90	24.00
13.00	16.70	43.55	25.90	34.90	25.60	33.05	27.60	31.05	27.00
13.00	17.90	43.80	30.00	34.95	24.40	33.10	26.60	31.10	26.60
13.10	16.20	45.00	28.20	35.05	22.10	33.15	24.90	31.10	26.80
13.15	15.40	32.00	24.20	35.10	25.70	33.20	26.50	31.10	27.10
13.15	15.60	32.25	26.00	35.15	23.90	33.25	23.60	31.20	23.20
13.15	18.40	36.30	24.50	35.35	23.50	33.30	24.30	31.20	23.40
13.20	15.00	36.40	26.10	35.35	28.80	33.35	27.00	31.35	21.00
13.20	17.00	36.75	26.40	35.40	24.60	33.40	24.50	31.40	24.60
13.30	17.20	36.85	24.60	35.40	24.60	33.45	20.50	31.45	24.60
13.40	17.50	36.90	25.00	35.40	25.40	33.60	26.00	31.60	26.20
13.45	15.50	37.25	26.80	35.60	24.90	33.75	25.00	31.60	26.30
13.45	17.30	37.35	21.10	35.60	25.70	33.80	28.70	31.60	26.60
13.45	18.80	37.35	27.50	35.65	24.10	33.85	24.80	31.65	21.20
13.50	18.40	37.50	26.40	35.65	25.60	33.85	28.30	31.65	24.10
13.55	16.70	37.50	27.30	35.80	24.10	34.05	24.70	31.70	25.20
13.55	18.00	37.90	27.10	35.80	25.50	34.40	23.80	31.70	25.50
13.60	15.00	37.90	28.00	35.90	21.00	31.85	27.30	31.75	26.40
13.60	18.40	37.90	29.00	35.95	26.30	31.90	24.40	31.80	23.10
13.65	16.80	36.15	25.70	35.95	26.90	31.90	25.50	31.80	26.60
13.70	16.10	13.85	18.20	13.85	17.20	13.80	16.80	13.75	15.50
13.75	14.60	29.50	26.40	29.80	25.60	29.90	22.60	30.00	25.60
29.70	21.30	29.55	21.60	29.85	23.30	29.90	24.50	30.00	26.40
29.75	24.00	29.60	24.90	29.85	25.50	29.90	25.30	30.00	27.50
29.75	24.20	29.60	26.60	29.85	26.40	30.00	25.40	14.00	18.40
29.80	22.70	29.65	24.40	14.00	17.00	29.70	20.60	14.10	18.00

29.80	25.20	29.65	25.20	14.15	17.50	14.15	17.70	14.15	19.70
14.20	12.10	30.05	22.50	28.55	17.90	27.30	24.20	22.15	22.20
14.20	17.30	30.05	26.20	28.55	23.30	27.30	24.60	22.15	23.30
14.20	18.20	30.10	24.80	28.55	25.30	27.30	25.10	22.15	24.90
14.20	19.20	30.10	24.90	28.55	25.50	27.35	22.70	22.20	13.40
14.30	20.90	30.15	24.80	28.55	22.00	27.35	23.30	22.20	20.40
14.35	18.00	30.20	21.70	28.60	23.40	27.35	24.40	22.20	21.70
14.35	20.50	30.25	25.20	28.65	23.50	27.40	23.90	22.20	22.00
14.45	16.80	30.25	26.30	28.65	24.60	27.40	24.10	22.20	23.90
14.45	18.80	30.25	26.70	28.75	22.90	27.40	24.20	22.20	22.90
14.45	20.10	30.30	22.30	28.75	24.50	27.40	25.00	22.35	15.20
14.45	21.00	30.35	24.70	28.75	25.60	27.40	25.10	22.35	21.40
14.55	13.40	29.40	24.80	28.75	26.10	27.40	25.90	22.40	18.50
14.55	19.10	29.40	25.30	28.75	26.40	27.50	23.60	22.40	22.00
14.60	14.70	29.40	26.00	28.75	26.50	27.65	23.00	22.40	22.00
14.60	17.80	29.45	22.00	28.90	26.30	27.65	23.30	22.40	23.00
14.60	18.00	29.45	24.50	28.95	24.40	27.70	20.70	26.60	25.60
14.60	18.70	29.45	25.70	28.95	25.20	27.75	26.30	26.65	23.70
14.60	19.90	29.50	22.00	29.00	22.60	27.85	22.90	26.65	23.80
14.60	20.80	15.10	19.50	29.05	22.00	27.85	23.30	26.70	18.20
14.65	18.40	15.15	18.70	29.05	23.70	27.85	25.10	26.70	21.60
14.70	16.10	28.30	22.00	29.05	23.90	27.85	25.60	26.70	24.20
14.70	17.80	28.35	21.90	29.05	24.20	27.85	26.30	26.70	24.20
14.70	19.00	28.35	23.20	29.05	24.60	27.90	24.30	26.75	19.50
14.75	16.40	28.35	26.50	29.05	26.60	27.90	24.70	26.75	22.00
14.75	19.50	28.35	26.90	29.15	25.00	27.95	19.80	26.75	22.70
14.75	19.50	28.40	19.00	29.15	25.40	27.95	22.60	26.75	23.50
14.75	20.00	28.40	22.00	29.15	26.00	27.95	25.50	26.85	20.30
14.95	15.20	28.40	24.70	29.20	24.40	28.00	23.70	26.90	22.20
14.95	20.90	28.45	26.40	29.20	24.50	28.05	17.00	26.90	23.80
14.95	21.00	28.50	25.10	29.25	25.40	28.05	24.80	26.90	23.90
15.00	18.70	27.15	23.70	29.30	23.90	28.05	25.50	26.90	24.00
15.00	21.60	27.20	20.40	29.30	26.60	28.00	22.20	26.90	24.10
15.05	16.90	27.20	21.00	29.35	18.40	28.15	25.00	26.95	21.90
15.10	17.20	27.20	23.00	29.35	25.20	28.15	26.10	26.95	24.30
15.10	17.50	27.20	24.00	29.40	20.70	28.20	22.80	27.00	23.30
15.10	19.30	27.20	25.50	29.40	22.40	28.25	23.70	27.05	25.70
15.10	19.30	27.25	21.80	27.10	24.60	28.30	19.60	27.10	21.20
20.80	22.70	20.85	18.60	20.90	21.40	20.95	20.10	21.00	20.80
20.80	23.60	20.85	24.60	20.90	24.10	20.95	20.60	21.00	21.20
20.85	17.80	20.90	21.10	20.95	18.00	20.95	22.50	21.05	20.50
21.05	22.00	21.80	18.30	20.20	21.70	19.55	22.20	18.75	22.10
21.05	22.80	21.80	18.90	20.25	21.40	19.60	12.00	18.90	17.90

21.10	19.70	21.80	20.90	20.25	22.20	19.60	22.00	18.90	20.30
21.10	20.40	21.80	22.50	20.30	17.20	19.60	22.50	18.90	20.70
21.10	22.60	21.80	23.80	20.30	19.60	19.65	17.80	18.90	20.80
21.15	22.10	21.80	24.10	20.30	20.50	19.65	20.30	18.95	19.00
21.20	22.20	21.85	14.00	20.30	22.00	19.65	20.40	19.00	21.80
21.20	22.60	21.85	20.70	20.30	24.50	19.65	21.00	19.00	22.20
21.25	19.40	21.85	21.40	20.35	22.80	19.70	20.50	19.05	18.40
21.25	21.70	21.85	21.60	20.40	16.90	19.70	20.80	19.05	19.00
21.30	19.00	21.85	22.10	20.40	19.10	19.70	21.00	19.05	21.30
21.35	19.00	21.85	23.90	20.40	22.70	19.70	21.20	19.05	21.70
21.35	23.10	21.90	21.30	20.45	14.00	19.70	21.30	19.10	19.00
21.40	18.80	21.90	22.10	20.45	18.00	19.70	22.70	19.10	20.00
21.40	20.10	21.90	22.40	20.45	18.90	19.70	23.00	19.10	20.00
21.40	21.00	22.00	20.00	20.45	19.00	19.75	18.40	19.10	21.00
21.40	22.00	22.00	20.70	20.45	20.40	19.75	21.00	19.10	22.40
21.40	22.90	22.00	22.60	20.45	21.00	19.75	22.20	19.15	18.50
21.40	23.10	22.00	23.30	20.45	23.50	19.75	22.80	19.15	20.50
21.45	21.70	22.05	23.10	20.50	20.90	19.75	22.90	19.15	20.50
21.45	21.90	22.05	23.50	20.50	21.90	19.80	19.10	19.15	22.30
21.50	15.10	22.10	21.70	20.50	22.20	19.80	21.50	19.20	17.20
21.50	20.00	22.10	22.90	20.50	22.30	19.85	17.00	19.20	20.10
21.50	20.80	22.10	23.70	20.50	22.30	19.85	21.10	19.20	21.50
21.50	22.20	22.15	14.70	20.50	23.20	19.85	21.80	19.20	22.00
21.50	22.80	22.15	20.30	20.50	23.70	19.90	17.30	19.20	22.50
21.50	22.90	16.45	18.00	20.55	19.30	19.90	20.00	19.25	13.50
21.55	19.90	16.45	18.10	20.55	22.90	19.90	21.30	19.25	18.40
21.55	20.70	16.45	19.90	20.55	23.20	19.95	22.20	19.25	21.80
21.55	21.10	20.70	21.50	20.60	22.30	19.95	22.50	19.25	22.90
21.55	22.60	20.70	21.60	20.65	18.80	20.00	19.30	19.30	19.40
21.55	23.70	20.70	21.60	20.65	19.10	20.00	21.80	19.30	21.40
21.60	22.40	20.70	21.80	20.65	24.00	20.00	22.40	19.30	22.60
21.60	24.70	20.70	23.00	20.70	19.00	20.00	23.80	19.35	22.00
21.65	14.90	20.75	20.20	20.70	20.40	20.05	19.80	19.35	22.90
21.65	20.40	20.75	23.80	20.70	21.20	20.05	20.30	19.35	25.70
21.65	21.30	20.75	24.50	20.70	21.50	20.05	21.00	19.40	13.20
21.65	21.80	20.80	17.20	19.45	19.70	20.10	16.30	19.40	19.60
21.70	22.90	20.15	20.80	19.45	20.80	20.15	19.00	19.40	19.70
19.40	20.60	20.20	19.90	19.45	21.50	20.15	20.70	19.40	19.80
19.45	17.60	20.20	21.00	19.46	22.90	19.50	19.40	19.40	20.00
19.50	19.70	25.15	22.40	25.55	24.20	26.30	23.00	24.30	24.70
19.50	19.80	25.20	19.70	25.60	21.20	26.30	23.30	24.35	23.50
19.50	21.20	25.20	20.50	25.60	23.80	26.30	23.30	24.35	24.20
19.50	21.90	25.20	22.50	25.60	24.30	26.30	23.900	24.40	20.90

19.50	22.20	25.20	23.10	25.60	24.50	26.30	23.90	24.40	22.60
19.50	22.40	25.20	24.70	25.65	22.60	26.30	24.90	24.45	19.30
19.50	22.40	25.20	24.90	25.70	20.50	26.35	19.60	24.45	19.40
19.50	23.00	25.25	18.50	25.70	22.60	26.35	22.10	24.45	21.70
25.00	23.40	25.25	21.30	25.75	18.00	26.35	24.00	24.45	22.00
25.00	23.50	25.25	24.30	25.75	23.50	26.40	22.70	24.45	24.90
25.00	24.30	25.30	26.40	25.75	24.60	26.40	27.10	24.50	20.50
25.00	25.20	25.35	21.10	25.80	20.20	26.50	23.10	24.50	22.40
25.05	21.50	25.35	21.70	25.80	24.50	26.50	25.80	24.50	23.20
25.10	20.40	25.35	24.90	25.80	26.80	26.55	19.80	24.55	20.80
25.10	20.80	25.35	25.90	25.85	23.80	26.55	22.50	24.55	20.90
25.10	22.00	25.35	25.90	25.85	23.90	26.60	23.90	24.55	23.10
25.15	21.10	25.40	20.20	25.85	24.90	45.05	27.40	24.55	23.90
16.50	15.90	25.40	22.40	25.85	25.00	48.40	29.20	24.70	21.70
16.50	17.80	25.40	22.50	25.95	22.80	22.40	23.20	24.70	23.20
16.50	20.40	25.40	23.80	25.95	23.40	18.35	19.20	24.75	24.10
16.50	22.50	25.40	24.20	25.95	24.20	18.35	20.40	24.75	24.70
16.60	18.30	25.45	19.40	25.95	25.10	18.35	21.50	24.75	25.20
16.60	19.70	25.45	21.70	26.00	22.90	18.35	21.90	24.75	25.60
16.60	21.00	25.45	23.40	26.00	22.90	18.35	22.20	24.80	22.40
16.65	16.40	25.45	23.60	26.00	23.70	18.40	17.00	24.80	24.60
16.70	18.30	25.45	23.60	26.05	22.10	18.40	18.80	24.80	25.10
16.70	18.90	25.45	25.80	26.05	23.70	18.40	19.20	24.85	22.70
16.70	19.70	25.50	21.70	26.05	26.70	18.40	20.40	24.85	23.30
16.80	19.20	25.50	22.70	26.10	17.80	18.40	21.80	24.85	23.60
16.80	21.40	25.50	23.50	26.15	20.60	18.40	22.00	24.85	24.40
16.85	17.50	25.50	23.90	26.20	22.50	18.40	22.60	24.85	26.20
16.85	19.00	25.55	17.00	26.20	27.00	18.45	18.90	24.90	17.40
16.85	19.50	25.55	21.90	26.25	22.30	18.45	18.90	24.90	18.60
16.90	19.10	25.55	22.20	18.65	18.40	18.45	19.40	24.90	20.00
16.90	19.40	25.55	22.40	18.65	19.90	18.45	19.50	24.90	21.60
16.90	20.00	25.55	23.30	18.65	20.20	18.50	20.10	24.90	23.20
16.90	20.40	25.55	23.40	18.65	22.50	18.50	21.70	24.95	18.00
16.90	21.00	17.00	21.10	18.70	17.00	18.50	24.10	24.95	20.50
16.95	14.60	17.05	16.60	17.10	19.10	18.60	19.00	24.95	23.60
17.00	18.30	17.05	19.20	17.10	20.30	18.60	21.40	24.95	23.70
17.00	19.30	17.05	19.60	17.10	21.10	17.15	17.10	25.00	21.20
17.00	19.80	17.05	20.70	17.15	16.80	17.15	17.40	25.00	21.60
17.20	20.00	22.50	23.80	23.05	20.40	23.55	19.50	22.50	20.70
17.20	20.10	22.55	23.30	23.05	22.30	23.55	20.60	22.50	21.90
17.20	21.00	22.55	23.30	23.05	23.30	23.55	21.10	22.50	22.00
17.20	22.10	22.60	23.40	23.10	22.00	23.55	23.30	22.50	23.00
17.95	18.00	22.60	23.50	23.10	22.30	23.55	23.90	17.45	23.7

17.95	19.70	22.60	24.80	23.10	23.10	23.55	23.90	17.50	18.50
17.95	19.80	22.65	18.80	23.10	23.80	23.60	16.40	17.50	19.80
17.95	21.10	22.65	22.60	23.15	18.20	23.60	22.50	17.50	20.10
17.95	21.20	22.65	22.80	23.15	20.70	23.60	23.30	17.50	20.80
18.00	12.80	22.65	24.50	23.15	23.60	23.65	21.00	17.55	14.70
18.00	18.00	22.70	21.00	23.15	24.80	23.65	23.10	17.55	19.20
18.00	21.10	22.70	21.30	23.20	22.30	23.70	21.60	17.60	12.50
18.00	21.80	22.70	23.00	23.20	23.80	23.70	23.30	17.65	18.10
18.05	18.50	22.75	17.70	23.25	18.30	23.75	21.30	17.70	18.10
18.05	18.60	22.75	19.50	23.25	21.70	23.80	22.00	17.70	19.80
18.10	18.10	22.75	21.40	23.25	22.60	23.80	23.20	22.45	24.10
18.10	18.60	22.80	22.20	23.25	22.90	23.80	25.70	22.50	13.60
18.10	19.40	22.80	22.70	23.25	23.00	23.85	20.20	17.85	22.00
18.15	19.80	22.80	23.70	23.25	25.30	23.85	20.60	17.90	19.40
18.15	21.00	22.85	21.30	23.30	21.10	23.85	22.70	17.90	21.50
18.20	18.20	22.85	22.50	23.30	21.40	23.85	22.70	17.95	18.00
18.20	20.50	22.85	23.30	23.30	21.90	23.85	23.20	22.50	19.00
18.20	21.00	22.90	17.00	23.35	21.50	23.85	24.40	22.50	20.60
18.25	18.10	22.90	20.30	23.35	22.50	23.90	23.30	22.45	22.80
18.25	21.50	22.90	20.80	23.35	22.80	23.90	23.40	17.85	21.20
18.30	18.00	22.90	23.10	23.40	21.40	23.95	22.70	17.90	20.40
18.30	18.50	22.90	23.30	23.40	22.30	23.95	23.60	24.30	23.70
18.30	19.90	22.90	26.00	23.40	22.50	23.95	24.50	22.45	21.80
18.30	20.10	22.95	22.50	23.40	22.70	24.00	22.50	17.85	20.20
18.30	20.60	22.95	22.60	23.40	24.40	24.05	18.20	23.50	23.70
18.30	21.20	22.95	23.20	23.45	16.30	24.05	18.50	24.30	23.50
18.30	21.40	23.00	21.70	23.45	21.20	24.05	22.30	22.45	21.40
18.35	17.80	17.70	21.40	23.45	21.30	24.05	22.90	17.80	22.70
18.35	18.20	17.70	22.30	23.45	21.40	24.05	25.00	23.50	23.50
52.30	29.50	17.75	18.50	23.45	23.00	24.10	22.20	24.30	21.30
52.90	26.40	17.75	19.20	23.45	23.00	24.10	23.20	22.45	21.00
57.25	26.00	17.75	20.50	23.50	19.10	24.10	23.40	17.80	20.10
22.45	15.90	17.75	20.80	23.50	20.60	24.25	22.80	23.50	22.60
22.45	20.60	17.80	19.60	23.50	22.60	24.25	23.20	24.30	21.00